Novel exact solutions for forced Boussinesq equation via extended generalized tanh-function method

by

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Approval Page

This is to certify that this undergraduate thesis entitled "Novel exact solutions for forced Boussinesq equation via extended generalized tanh-function method" submitted by Ralph Jay P. Torres to fulfill part of the requirements for the degree of Bachelor of Science in Physics was successfully defended and approved on the 24th of May 2025.

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Abstract

This thesis work introduces a novel extended generalized tanh-function method for deriving exact solutions to nonlinear partial differential equations. Central to this approach is an ansatz Y_p incorporating a tunable parameter p which provides significant flexibility in the characteristics of the resulting solution families. The method is applied to the classical Boussinesq equation.

Application of this extended method yields 8 unique families of exact, tunable solutions, including solitons, non-soliton traveling waves, and plane periodic solutions. Critically, for $p \neq 1$, these solutions pertain to a forced Boussinesq equation with the forcing term $F(Y_p)$ explicitly dependent on p. Solutions to the original, unforced Boussinesq equation, as obtained through standard tanh method, are recovered when $p = 1$ is set and where $F(Y_p)$ vanishes.

The parameter p is found to greatly influence solution characteristics. For $0 \leq p \leq 1$, localized waves generally widen and flatten. For $p > 1$, they narrow and heighten. A fundamental transformation to trigonometric forms and oscillatory behavior occurs for $p < 0$, where the wave number becomes imaginary, potentially introducing singularities.

This work significantly expands the analytical solution space for Boussinesq-type equations, demonstrating the method's capacity to generate a diverse spectrum of wave behaviors. The study underscores the importance of the tunable parameter p and the associated forcing function, opening new avenues for theoretical modeling and understanding nonlinear wave phenomena. Future research includes applying the method to other nonlinear systems and further exploring the parameter space and physical implications.

Keywords: nonlinear partial differential equations, Boussinesq equation, generalized tanh method, extended generalized tanh method, tunable solutions, solitary waves, solitons, periodic waves, forcing function

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List of Abbreviations

cas computer algebra system

g-hath generalized half-angle tanh

ode ordinary differential equation

pde partial differential equation

wrt with respect to

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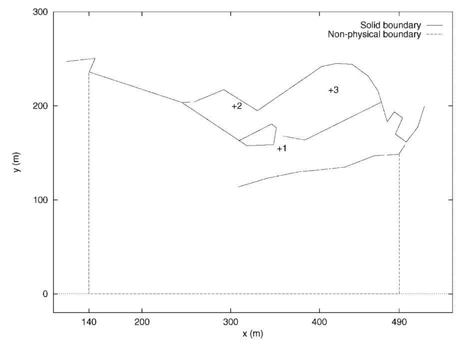
1. Introduction

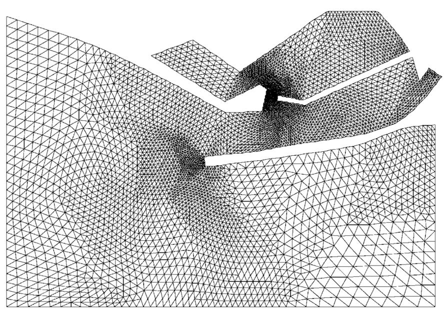
The study of nonlinear wave phenomena represents one of the most challenging and significant areas in mathematical physics. Among the many nonlinear partial differential equations that describe such phenomena, the Boussinesq equation stands out for both its physical significance and mathematical richness. This equation, which in dimensionless form is given by

$$\partial_t^2 u - c^2 \partial_x^2 - \alpha \partial_x^2 u^2 - \beta \partial_x^4 u = 0, \tag{1.1}$$

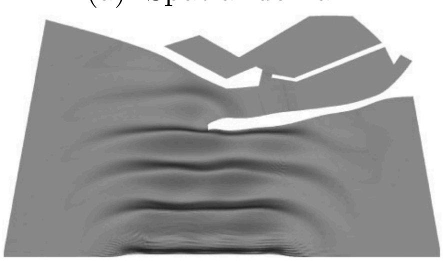
has shaped our understanding of long waves in shallow water since its introduction by Boussinesq in the 1870s [8,9] (see also [10]). While it found its primary application in hydrodynamics and coastal engineering, such as in modeling wave interactions in various nearshore zones as illustrated in Figure 1, this equation has proven surprisingly versatile. It appears in wide-ranging physical systems, including nonlinear magnetosound waves in plasmas [11,12], observed thin turbulent layers in the atmosphere [13,14], nonlinear wave perturbations in acoustic-like regimes [15], electromagnetic waves in nonlinear dielectrics [16], elastic waves in antiferromagnets [17], and vibrations in nonlinear strings [18].



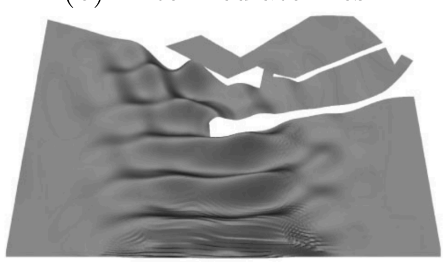
(a): Spatial domain



(b): Intermediate mesh



(c): Model at $t = 30$



(d): Model at $t = 30$

Figure 1: Unstructured triangular meshes of the harbor geometry (a, b) and simulated model of the the waves on the free surface at time t (c, d) [1].

The Boussinesq equation derives from a family of nonlinear equations characterized by a second-order time derivative $\partial_t^2 u$ and the general form

$$\partial_t^2 u - \partial_x^2 u + P(u) = 0, \tag{1.2}$$

where $P(u)$ is a nonlinear term, and $u = u(x,t)$ is a differentiable function of space x and time t. Unlike unidirectional pdes such as the Korteweg-de Vries (KdV) and KdV-type equations which involve a $\partial_t u$ term, the Boussinesq equation exhibits bidirectional wave propagation, traveling in both left and right directions [19]. However, despite this distinction, this equation is actually closely related to other key equations in nonlinear wave theory. For instance, it reduces to the KdV equation if the interaction of opposing waves is neglected, considering only one direction [20]. It can also be obtained from or reduced to the Kadomtsev-Petviashvili (KP) equation under certain conditions [21,22]. Furthermore, it can approximate the nonlinear Schrödinger (NLS) equation for complex-valued amplitudes in the slow modulation regime, with its rational solutions bearing resemblance to the NLS rogue waves. A crude overview of these relationships and applications is presented in Figure 2.

The equation incorporates competing effects: nonlinearity, which steepens the wave, and linear dispersion, which spreads it [9]. This balance allows for the existence of soliton solutions, which are particle-like waves with a stable profile and constant shape and speed [23]. The Boussinesq equation also accounts for frequency dispersion, enabling it to model a wider range of wave phenomena than classical shallow-water equations, those derived from the Navier-Stokes equations say [24]. However, soliton solutions specific to the Boussinesq equation can exhibit complex behaviors like singularity formation or decay under perturbations [21,25,26] (see also [27]). Additionally, different forms of the equation exist. For instance, depending on the sign of β, the equation can be ill-posed when $\beta = 1$ or well-posed when $\beta = -1$, though both classical forms can be completely integrable [23,28]. The improved Boussinesq equations, which modify the dispersive term by incorporating a mixed fourth-order derivative $\partial_t^2 \partial_x^2 u$

instead of the purely spatial ∂_x^4 term [29,30], are also studied but are not the focus of this work.

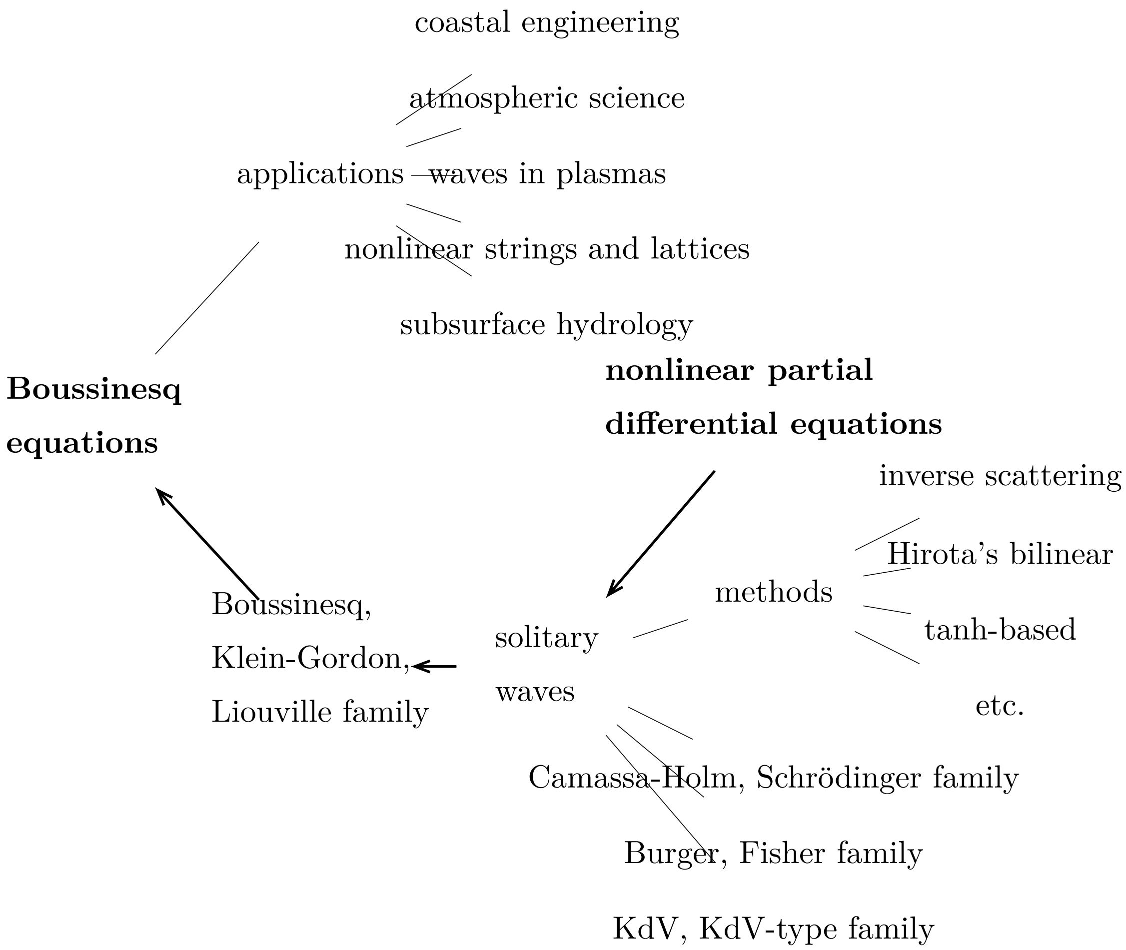


Figure 2: Overview of the Boussinesq equation: relationships to other solitary wave-describing nonlinear pdes, solution techniques, and applications.

While various analytical methods have been developed to solve the Boussinesq equation and other solitary wave-describing nonlinear pdes, traditional approaches like the standard tanh-function method often produce solutions with fixed characteristics, limiting their flexibility for modeling diverse physical phenomena. This thesis addresses this limitation by developing and applying a more adaptable solution method that can produce tunable solution families based on an extension and generalization of the widely-used standard tanh-function method.

In this work, we introduce and systematically apply our extended generalized tanh-function method to derive exact solutions for the classical Boussinesq equation, specifically the form with $\alpha = 3$ and $\beta = 1$, often referred to in literature. Central to this approach is an ansatz incorporating a tunable parameter p, offering significant flexibility in the characteristics of the resulting solution families. This thesis will detail the formulation of this generalized method and a subsequent novel extension designed to further expand the solution space.

By applying these methods, this study aims to derive new tunable soliton, periodic, and other traveling wave solutions. A key aspect will be demonstrating how these solutions, particularly for $p \neq 1$, pertain to a forced version of the Boussinesq equation, where the forcing term is explicitly dependent on the tunable parameter p. It will also be shown that solutions to the original, unforced Boussinesq equation, as obtainable through the standard tanh method, can be recovered as special cases. The influence of the tunable parameter p on solution characteristics will be a central point of discussion.

The remainder of this thesis is organized as follows: Section 2 provides a comprehensive review of the Boussinesq equation and the tanh-function method, Section 3 details the mathematical foundations of our extended generalized tanh-function method. Section 4 systematically applies this method to the Boussinesq equation deriving and analyzing new solution families, and explores the physical interpretations of these solutions and the significance of the tunable parameter p. Finally, Section 5 summarizes our contributions and discusses directions for future research.

This paper focuses on the development and application of this generalized method to the $(1+1)$-dimensional Boussinesq equation. While not intended to improve the Boussinesq equation itself, the derived forcing functions and the tunable nature of the solutions may offer new perspectives for physical modeling. This work lays the groundwork for a robust generalization of the tanh-function method, with potential future research directions including its application to other nonlinear systems and further exploration of the parameter space and forcing functions.

2. Review of Related Literature

2.1. Boussinesq equation in depth

The classical Boussinesq equation, expressed in dimensionless form as

$$\partial_t^2 u - c^2 \partial_x^2 - \alpha \partial_x^2 u^2 - \beta \partial_x^4 u = 0 \tag{2.1}$$

has been extensively studied since its original formulation by Boussinesq in the 1870s for describing long waves in shallow water [8,9]. Ursell [10] provided important early theoretical developments that established the equation's mathematical foundation.

The equation's applicability extends far beyond its original hydrodynamic context. Karpman [11], followed by Scott [12], demonstrated its relevance to nonlinear magnetosound waves in plasma physics. In atmospheric sciences, Klein [13] and Achatz [14] identified the equation's role in modeling thin turbulent layers. Whitham [15] explored its applications to nonlinear wave perturbations in acoustic regimes, while Xu, Auston and Hasegawa [16] investigated electromagnetic wave phenomena in nonlinear dielectrics. Further applications include elastic wave propagation in antiferromagnets [17] and vibrations in nonlinear string systems [18].

Wazwaz [19] provides a comprehensive analysis of the equation's bidirectional wave propagation properties, contrasting it with unidirectional equations such as the Korteweg-de Vries equation. The mathematical connections between these equations have been thoroughly investigated. Korteweg and de Vries [20] established the relationship whereby the Boussinesq equation reduces to the KdV equation under unidirectional assumptions. Bogdanov and Zakharov [21] demonstrated how the equation can be derived from the Kadomtsev-Petviashvili equation through dimensional reduction techniques, while Chen [22] showed the reverse relationship under near-unidirectional conditions. Clarkson and Dowie [31] investigated the connections to the nonlinear Schrödinger equation, particularly in the context of rational solutions and their relationship to rogue wave phenomena. This work builds on earlier studies of extreme wave events by Dysthe, Krogstad and Müller [32], and Kharif, Pelinovsky and Slunyaev [33], establishing the Boussinesq equation's relevance to understanding unpredictable wave behavior.

The equation's treatment of competing physical effects has been well-documented in the literature. Boussinesq's original work [9] identified the balance between nonlinear steepening and linear dispersion as fundamental to the equation's wave-describing capabilities. Dingemans [24] provided detailed analysis of how frequency dispersion in the Boussinesq framework enables modeling of shorter wavelength phenomena compared to classical shallow-water approaches derived from Navier-Stokes equations.

The equation's soliton solutions have also received considerable attention. Hereman [23] provides comprehensive analysis of particle-like wave solutions, characterized by their single-humped profiles and constant propagation properties. However, several studies have identified anomalous behaviors in Boussinesq solitons. Bogdanov and Zakharov [21], Yang and Wang [25], and Kutev et al. [26] documented finite-time singularity formation and perturbation-induced decay phenomena. Earlier work by Falkovich, Spector and Turitsyn [27] provided theoretical foundations for understanding these instability mechanisms.

The mathematical well-posedness of different Boussinesq forms has been thoroughly investigated. Clarkson and Kruskal [34] established general principles for coefficient scaling and sign considerations, demonstrating that equations with $\alpha > 0$ and varying β values yield equivalent forms under appropriate transformations. McKean [28] provided crucial analysis showing that $\beta = -1$ leads to well-posed formulations, while Hereman [23] confirmed that $\beta = 1$ results in ill-posed problems for arbitrary initial data. Despite posedness considerations, both classical forms maintain complete integrability, as demonstrated by Zakharov [18] and McKean [28]. This complete integrability, involving infinite conservation laws and symmetries, places the Boussinesq equation among the select few completely integrable nonlinear partial differential equations. Ablowitz and Segur [35] provide comprehensive treatment of the implications of complete integrability, including multi-soliton solution existence and inverse scattering formalism applicability.

Recent developments have focused on improved Boussinesq formulations incorporating mixed temporal-spatial derivatives. Bona, Chen and Saut [29,36] established well-posedness results for equations featuring $\partial_t^2 \partial_x^2 u$ terms instead of purely spatial fourth-order derivatives. Hereman [23] demonstrated how these modifications enhance dispersive properties and broaden applicability to diverse wave phenomena. However, Christov, Maugin and Porubov [30], and Madsen, Murray and Sørensen [37] showed that improved formulations sacrifice complete integrability, though they maintain well-posedness and offer enhanced modeling capabilities for practical applications.

2.2. Standard tanh method and others

A variety of methods have been developed to solve the Boussinesq equation and other solitary wave-describing families of nonlinear partial differential equations. These include powerful techniques that directly deal with the partial differential equations such as inverse scattering transform [38], Bäcklund transform [39], and Hirota's bilinear method [40–42]. However, simpler methods such as direct integration [43], homogeneous balance method [44], sine-Gordon expansion [45,46], and tanh-function method [2], [3] have also proven effective in obtaining exact and analytic solutions. These methods capitalize on the straightforward nature of hyperbolic and exponential functions to model traveling waves, and of trigonometric functions to represent periodic waves, which solitary wave-describing equations readily accommodate. By adopting a traveling wave frame of reference, the partial differential equation is transformed into an ordinary differential equation from which closed-form solutions in terms of these transcendental functions are sought [19,23].

Due to simplicity, the original tanh-based method has since been extended and modified in certain directions to obtain more exact traveling wave solutions. This includes the coth extension [47,48], hyperbolic-function generalization [49,50], trigonometric [19,51] and exponential [52] reformulations, and generalizations to Riccati equation expansion [53,54] and projection [55–58]. The lattermost method can obtain new families of exact solutions including non-traveling wave soliton-like ones among others.

Building upon the solution methods discussed previously, the standard tanh-function method introduced by Malfliet in @malfliet1996-1 @malfliet1996-2 emerges as a particularly accessible technique that has become foundational to modern analytical approaches for nonlinear wave equations. This method involves transforming the pde into a nonlinear ode through the transformation

$$\begin{aligned} &u(x,t) \to U(\xi), \\ &\xi = \mu(x - ct), \end{aligned} \tag{2.2}$$

where c and μ represent arbitrary real constants typically interpreted as wave speed and wave number, respectively, and introduces the function

$$Y(\xi) = \tanh \xi. \tag{2.3}$$

This function was specifically chosen for its self-similarity in the context of differentiation. That is, when differentiated repeatedly, the tanh functions assume the form of slight variations of itself and transforms to sech quite easily as in

$$\begin{aligned} &\mathrm{d}_\xi Y = \operatorname{sech}^2 \xi = 1 - Y^2, \\ &\mathrm{d}_\xi^2 Y = -2Y + 2Y^3, \\ &\mathrm{d}_\xi^3 Y = -2 + 8Y^2 - 6Y^4, \\ &\qquad \vdots \end{aligned} \tag{2.4}$$

allowing us to derive a set of algebraic functions U that represent various orders of derivatives

$$U = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{2.5}$$

which will serve as the solution to the pde to which we are applying this method. Here, the coefficients a_k are real constants to be determined, and M is a positive integer that can be extracted by balancing terms and derivatives.

The standard tanh method encompasses the following steps:

1. transforming the pde into a nonlinear ode,
2. solving the resulting derivatives,
3. balancing the highest order nonlinear term with the highest order derivative,
4. deriving and solving a nonlinear system of equations for coefficients and parameters, and
5. substituting the solutions for these coefficients and parameters back into the nonlinear ode.

2.3. Recent works on generalized tanh method

Recent studies have explored generalization of the standard tanh method to address limitations when applied to forced or inhomogeneous nonlinear pdes. Domingo and Dingel [5,59] continued formulating the generalized half-angle tanh (g-hath) ansatz, which is exactly the introductory function Y_p that we have, demonstrating its application to forced versions of the Huxley equation and showing how solutions with tunable parameter p reduce to standard tanh solutions when $p = 1$. This approach has been successfully extended to other equations: Parel and Dingel [6] applied the method to inhomogeneous Burgers-Fisher equations, Hao and Dingel [7] addressed forced Sine-Gordon equations, and Bayan and Dingel [60] tackled inhomogeneous Fisher equations the generalized approach. Most recently, Cornista and Dingel is reportedly proposing further modifications to this method for polynomial nonlinear evolution equations, obtaining solutions that cannot be derived through standard tanh or tanh-coth methods when applied to the Benjamin-Bona-Mahony equation.

Despite these advances, existing generalizations have primarily employed truncated series expansions using only positive powers of the ansatz function Y, which limits the solution space to tanh and sech functions (and their corresponding trigonometric analogues). This constraint restricts the diversity of obtainable wave solutions. This work extends these generalizations by incorporating both positive and negative powers in the series expansion, thereby broadening the solution space to include coth and csch as well as their combinations, providing a more comprehensive framework for generating diverse solution families to classical integrable equations like the Boussinesq equation.

3. Methodology

In this section, we present our proposed generalization of the tanh-function method and a novel extension of this generalization. These methods are developed to obtain new tunable soliton, periodic, and other traveling wave solutions for forced Boussinesq equation.

As discussed in Section 2, we build upon the tanh-function method due to its inherent simplicity and capacity to generate a diverse range of solutions. The standard tanh method, while primarily limited to single-soliton solutions unlike methods that produce multiple-soliton solutions, offers adaptability and ease of implementation that make it an ideal foundation and starting point for extensions. Furthermore, the clear algebraic structure arising from tanh derivatives simplifies the process of finding and classifying solutions.

3.1. Generalization of the tanh method

Building upon the standard tanh-function approach, we replace the traditional introductory function Y with a novel ansatz first presented by Buenaventura, Dingel and Calgo in [4], inspired by the half-angle identity in tanh-function and parametrized by a tunable parameter p

$$Y_{p,\xi} = Y_p(\mu\xi) = (1+p)\frac{\tanh\dfrac{\mu\xi}{2}}{1+p\tanh^2\dfrac{\mu\xi}{2}}, \qquad 0 \leq p \leq 1, \quad p \in \mathbb{R}. \tag{3.1}$$

The key feature of this ansatz is the tunable parameter p. It could allow for solutions to be either adaptively tailored to the specific problem at hand or precisely fine-tuned to meet specific conditions. Following Malfliet's approach outlined in Figure 3, we transform the pde

$$p(u, \partial_t u, \partial_x u, \partial_t^2 u, \partial_x^2 u, \partial_x\partial_t u, \ldots) = 0 \tag{3.2}$$

into a nonlinear ordinary differential equation

$$P\Big(U, \mathrm{d}_\xi U, \mathrm{d}_\xi^2 U, \ldots\Big) = 0 \tag{3.3}$$

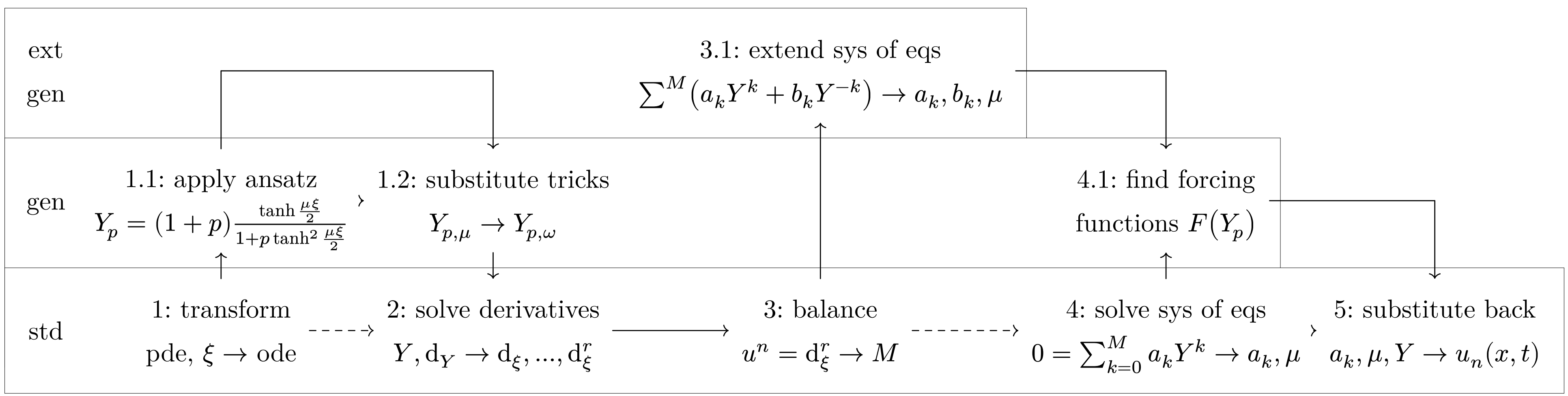


Figure 3: Procedures of the standard (std) tanh method [2,3], along with generalization (gen) [4–7], and subsequent novel extension (ext gen) of the generalization.

together with their respective solutions $u(x,t)$ and $U(\xi)$ using the variable

$$\xi = x - ct. \tag{3.4}$$

Assuming the integration constants vanish, we iteratively integrate this ode until the desired order is achieved, say until

$$\int \cdots \int P\Big(U, \mathrm{d}_\xi U, \mathrm{d}_\xi^2 U, \mathrm{d}_\xi^3 U, \ldots; Y\Big) = 0, \tag{3.5}$$

as long as all terms retain derivatives. We then compute for the higher-order derivatives

$$\mathrm{d}_\xi, \mathrm{d}_\xi^2, \mathrm{d}_\xi^3, \ldots, \mathrm{d}_\xi^n \tag{3.6}$$

with the highest order n present in the integrated ode in (3.5). Note that this computation is particularly cumbersome.

Next, we assume that the series

$$U = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{3.7}$$

remains admissible as a solution under this generalized tanh method, allowing

$$u(x,t) = U(\xi) = S(Y) \tag{3.8}$$

to also be a solution to the ode. We balance the highest order nonlinear term with the highest order derivative following the mappings

$$\begin{aligned} &u \to M, \quad u^2 \to 2M, \quad \ldots, \quad u^n \to nM; \\ &\partial u \to M+1, \quad \partial^2 u \to M+2, \quad \ldots, \quad \partial^r u \to M+r. \end{aligned} \tag{3.9}$$

We employ this to balance the highest order nonlinear term with the highest order derivative in the integrated ode in (3.5) and determine the balance constant M to use in (3.7). We reject any non-positive integer M and adjust in the integration step accordingly. If inconvenient, we apply the transformation

$$U(\xi) = \varphi^M \xi, \tag{3.10}$$

then substitute it back and attempt to determine M again as long as M is a fraction or a negative integer as suggested in [57].

We then substitute the computed derivatives and the series in (3.7) with the determined M into the integrated ode, grouping terms according to their powers in Y. For terms with non-integral powers of Y, we introduce forcing functions $F(Y)$ to eliminate them resulting in

$$\int \cdots \int P\Big(U, \mathrm{d}_\xi U, \mathrm{d}_\xi^2 U, \mathrm{d}_\xi^3 U, \ldots; Y\Big) = F(Y). \tag{3.11}$$

This transforms our ode, and by extension the pde, into a forced version. To be consistent for all values of Y, the coefficient expressions must each equate to zero. This results in a nonlinear system of algebraic equations for the mathematical coefficients a_n for $n \geq 0$, $n \in \mathbb{Z}$ and physical coefficients such as the wave number μ. We then solve this system by hand, and utilize a computer algebra system such as the free and open-source Sage for tedious calculations when needed.

Finally, we substitute the determined solutions for the coefficients and parameters back into the integrated ode, apply restricting conditions where necessary, and obtain a set of tunable soliton and plane periodic solutions.

3.2. Extension of the generalized tanh method

In the previous method, we only have algebraic terms in positive powers of Y in the finite series in (3.7), which restricted the solution space to tanh and sech-based solutions. To explore a broader set of solutions, particularly those based on coth and csch, we extend the series to

$$S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{3.12}$$

as inspired by an extension of the tanh method presented in [47,48]. To the best of our knowledge, this specific method has not been previously reported in the literature.[1]

[1] A paper detailing initial results of the application of this method has been submitted to the proceedings of the 43rd Samahang Pisika ng Pilipinas Physics Conference (June 2025) but is currently under review.

3.3. Application to the Boussinesq equation

After formulating our proposed generalization of the tanh-function method along with its extension, we implement both methods to obtain new tunable solutions to the classical form of the Boussinesq equation with $\alpha = 3$ and $\beta = 1$ [8,9,23]

$$\partial_t^2 u - c^2 \partial_x^2 u - \alpha \partial_x^2 u^2 - \beta \partial_x^4 u = 0. \tag{3.13}$$

The detailed application of these methods to derive specific solution families is presented in the following chapter, where we systematically work through the computational steps and analyze the resulting solutions.

4. Results and Discussion

This chapter presents our findings from applying our novel generalized tanh-function method and its extension to the classical Boussinesq equation. We report the derivation of new tunable exact solutions, including soliton, non-soliton traveling wave, and plane periodic solutions. We also analyze the tunable parameter p introduced in our methods and discuss its impact on the characteristics of the solutions.

4.1. Solutions via standard tanh method

We first addressed the classical Boussinesq equation given by

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2(3u^2) - \partial_x^4 u = 0 \tag{4.1}$$

using the standard tanh method. Following the procedure outlined in Section 3, we transformed this nonlinear pde to a nonlinear ode through the transformation $\xi = \mu(x - ct)$ yielding

$$c^2 \mathrm{d}_\xi^2 u - \mathrm{d}_\xi^2 u - 3\mathrm{d}_\xi^2 u^2 - \mathrm{d}_\xi^4 u = 0 \tag{4.2}$$

where c and μ represent wave speed and wave number, respectively. We then iteratively integrated this ode twice wrt ξ as follows

$$\begin{aligned} &(c^2 - 1)\mathrm{d}_\xi^2 u - 3\mathrm{d}_\xi^2 u^2 - \mathrm{d}_\xi^4 u = 0 \\ \implies &(c^2 - 1)\mathrm{d}_\xi u - 3\mathrm{d}_\xi u^2 - \mathrm{d}_\xi^3 u = 0 \\ \implies &(c^2 - 1)u - 3u^2 - \mathrm{d}_\xi^2 u = 0. \end{aligned} \tag{4.3}$$

To determine the balance coefficient, we balanced the highest-order nonlinear term with highest-order derivative following the previously discussed mapping and obtained

$$\begin{aligned} &u^2 = \mathrm{d}_\xi^2 u \\ \implies &2M = M + 2 \\ &M = 2. \end{aligned} \tag{4.4}$$

Recall that this method admits the use of the series substitution

$$u(x,t) = S(Y) = \sum_{i=0}^{M} a_i Y_i$$

$$\Longrightarrow U(x,t) = a_0 + a_1 Y + a_2 Y^2 \tag{4.5}$$

as solution to the ode, and consequently the pde, based on the new independent variable

$$Y(\xi) = \tanh \xi. \tag{4.6}$$

This ansatz was introduced by Malfliet in [2,3]. We computed the first and second derivatives wrt ξ and got

$$\begin{aligned} \mathrm{d}_\xi &= \mathrm{d}_\xi Y \cdot \mathrm{d}_Y \\ &= \mathrm{d}_\xi \tanh \mu\xi \cdot \mathrm{d}_Y \\ &= \mu(1 - Y^2) \cdot \mathrm{d}_Y \end{aligned} \tag{4.7}$$

since

$$\begin{aligned} \mathrm{d}_\xi \tanh \mu\xi &= \mu \operatorname{sech}^2 \mu\xi \\ &= \mu(1 - \tanh^2 \mu\xi) \\ &= \mu(1 - Y^2), \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \mathrm{d}_\xi^2 &= \mathrm{d}_\xi \cdot \big(\mathrm{d}_\xi Y \cdot \mathrm{d}_Y\big) \\ &= \mu(1 - Y^2) \cdot \mathrm{d}_Y [\mu(1 - Y^2) \cdot \mathrm{d}_Y] \\ &= \mu(1 - Y^2) [\mu(1 - Y^2) \cdot \mathrm{d}_Y^2 + \mu(0 - 2Y) \cdot \mathrm{d}_Y] \\ &= -2\mu^2 Y (1 - Y^2) \cdot \mathrm{d}_Y + \mu^2 \big(1 - Y^2\big)^2 \cdot \mathrm{d}_Y^2. \end{aligned} \tag{4.9}$$

Substituting this derivative and the series into the ode, and grouping terms by powers of Y, we obtained

$$\begin{aligned} 0 = {} & (c^2 - 1)(a_0 + a_1 Y + a_2 Y^2) \\ & -3\big(a_0 + a_1 Y + a_2 Y^2\big)^2 \\ & -\Big(-2\mu^2 Y (1 - Y^2) \cdot \mathrm{d}_Y + \mu^2 \big(1 - Y^2\big)^2 \cdot \mathrm{d}_Y^2\Big)(a_0 + a_1 Y + a_2 Y^2) \\ = {} & (c^2 - 1)a_0 \ + (c^2 - 1)a_1 Y + (c^2 - 1)a_2 Y^2 \end{aligned}$$

$$
\begin{array}{lllll}
-3[a_0^2 & + 2a_0a_1Y & + 2a_0a_2Y^2 + 2a_1a_2Y^3 & + a_2^2Y^4 & \\
 & & +a_1^2Y^2 & &] \\
-(-2\mu^2)[& a_1Y & + 2a_2Y^2 & - a_1Y^3 - 2a_2Y^4] & \\
-\mu^2[2a_2 & & - 4a_2Y^2 & + 2a_2Y^4] &
\end{array}
$$

$$
\begin{aligned}
&= [(c^2-1)a_0 - 3a_0^2 - 2\mu^2a_2] \\
&\quad +Y[(c^2-1)a_1 - 6a_0a_1 + 2\mu^2a_1] \\
&\quad +Y^2[(c^2-1)a_2 - 6a_0a_2 + 8\mu^2a_2 - 3a_1^2] \\
&\quad +Y^3[-6a_1a_2 - 2\mu^2a_1] \\
&\quad +Y^4[-3a_2^2 - 6\mu^2a_2]. \qquad (4.10)
\end{aligned}
$$

Equating the coefficients of each power of Y to 0, we formed the following system of equations

$$
\begin{aligned}
Y^0 &: \quad 0 = (c^2-1)a_0 - 3a_0^2 - 2\mu^2a_2 \\
Y^1 &: \quad 0 = (c^2-1)a_1 - 6a_0a_1 + 2\mu^2a_1 \\
Y^2 &: \quad 0 = (c^2-1)a_2 - 6a_0a_2 + 8\mu^2a_2 - 3a_1^2 \\
Y^3 &: \quad 0 = -6a_1a_2 - 2\mu^2a_1 \\
Y^4 &: \quad 0 = -3a_2^2 - 6\mu^2a_2, \qquad (4.11)
\end{aligned}
$$

which we then solved to compute the coefficients needed for the ode. Solving by hand, we obtained the set of solutions

$$
\begin{aligned}
y_0 &: \quad \mu = m \in \mathbb{R}, \quad a_0 = 0, \frac{1}{3}(c^2-1), \quad a_1 = 0, \quad a_2 = 0 \\
y_1 &: \quad \mu = \pm\frac{1}{2}\sqrt{c^2-1}, \quad a_0 = \frac{1}{2}(c^2-1), \quad a_1 = 0, \quad a_2 = \frac{1}{2}(1-c^2) \\
y_2 &: \quad \mu = \pm\frac{1}{2}\sqrt{1-c^2}, \quad a_0 = \frac{1}{6}(1-c^2), \quad a_1 = 0, \quad a_2 = \frac{1}{2}(c^2-1). \qquad (4.12)
\end{aligned}
$$

Note that some solutions were paired, with with parity being the only distinguishing characteristic. Substituting these solutions back into the nonlinear pde yielded physically realizable results.

Specifically, while y_0 gives trivial solutions, y_1 and y_2 where $c^2 > 1$ (supercritical wave speed) give the soliton solutions

$$
\begin{aligned}
u_1(x,t)_{\text{std}} &= \frac{c^2-1}{2}\left[1-\tanh^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] \\
&= \frac{c^2-1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) \qquad (4.13)
\end{aligned}
$$

$$
u_2(x,t)_{\text{std}} = -\frac{c^2-1}{6}\left[1-3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]. \qquad (4.14)
$$

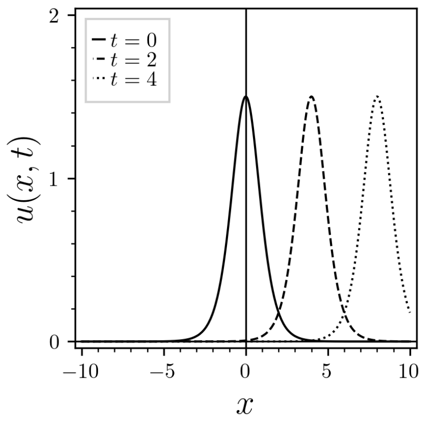
In the opposite regime, where $c^2 < 1$ (subcritical wave speed), y_1 and y_2 give the plane periodic solutions

$$
\begin{aligned}
u_3(x,t)_{\text{std}} &= \frac{c^2-1}{2}\left[1+\tan^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right] \\
&= \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right) \qquad (4.15)
\end{aligned}
$$

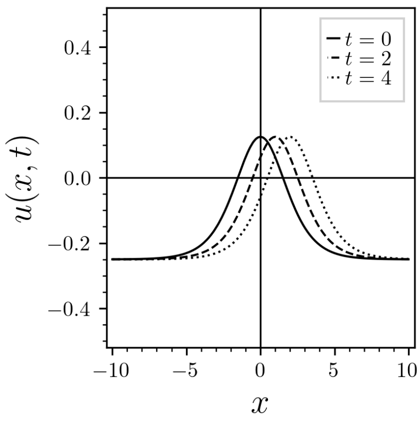
$$
u_4(x,t)_{\text{std}} = -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]. \qquad (4.16)
$$

We note that u_1, u_2, u_3, u_4 correspond to the solutions found in [19]. These solutions, plotted in Figure 4, represent the baseline results obtainable with the most fundamental tanh-function approach. Solutions u_1 and u_2 are the classic bell-shaped soliton, a localized wave of elevation maintaining its shape. Solutions u_3 and u_4 represent a train of periodic waves with singularities. These forms are well-documented and serve as our benchmark.

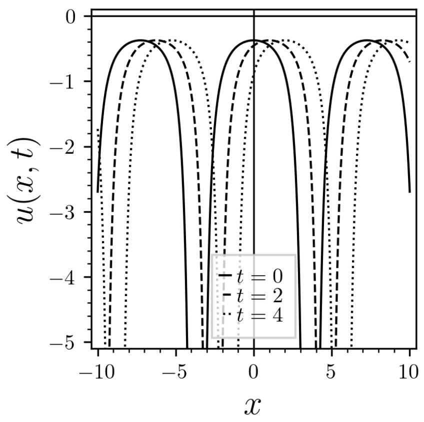
The standard tanh method, while effective for finding these fundamental solutions, is limited because the solution forms are fixed once the balance coefficient M is determined. It does not offer inherent tunability beyond the wave speed c.



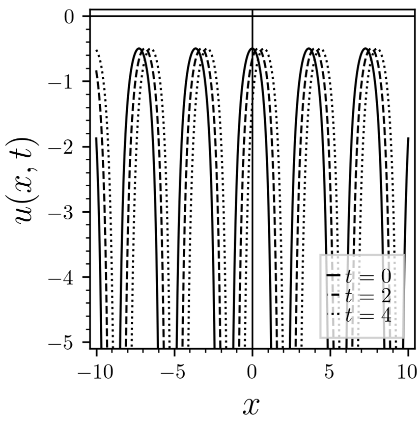
(a): $u_{1,\text{std}} : c = 2, |x| \le 10$



(b): $u_{2,\text{std}} : c = \frac{1}{2}, |x| \le 10$



(c): $u_{3,\text{std}} : c = 2, |x| \le 10$



(d): $u_{4,\text{std}} : c = \frac{1}{2}, |x| \le 10$

Figure 4: Plots of the solutions to the classical Boussinesq equation via standard tanh method, with $t = 0, 2, 4$.

4.2. Solutions via extended standard tanh method

To explore a broader solution space, we employed the extended standard tanh method. This involves using a longer series expansion

$$
\begin{aligned}
& u(x,t) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k} \\
& \Longrightarrow U(x,t) = a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}
\end{aligned} \tag{4.17}
$$

which incorporates terms with negative powers of Y giving us

$$
\begin{aligned}
0 = (c^2 - 1)&\big(a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}\big) \\
&-3\big(a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}\big)^2
\end{aligned}
$$

$$-\Big(-2\mu^2 Y(1-Y^2)\cdot \mathrm{d}_Y + \mu^2 (1-Y^2)^2 \cdot \mathrm{d}_Y^2\Big)$$
$$(a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}).$$

This extension allows for solutions involving coth and csch functinos, in addition to tanh and sech functions. The resulting system of equations is larger and contains more unknown coefficients

$$\begin{aligned}
Y^{-4}:&\quad -6b_2\mu^2 - 3b_2^2 = 0\\
Y^{-3}:&\quad -2b_1\mu^2 - 6b_1 b_2 = 0\\
Y^{-2}:&\quad b_2c^2 + 8b_2\mu^2 - 3b_1^2 - 6a_0b_2 - b_2 = 0\\
Y^{-1}:&\quad b_1c^2 + 2b_1\mu^2 - 6a_0b_1 - 6a_1b_2 - b_1 = 0\\
Y^{0}:&\quad a_0c^2 - 2a_2\mu^2 - 2b_2\mu^2 - 3a_0^2 - 6a_1b_1 - 6a_2b_2 - a_0 = 0\\
Y^{1}:&\quad a_1c^2 + 2a_1\mu^2 - 6a_0a_1 - 6a_2b_1 - a_1 = 0\\
Y^{2}:&\quad a_2c^2 + 8a_2\mu^2 - 3a_1^2 - 6a_0a_2 - a_2 = 0\\
Y^{3}:&\quad -2a_1\mu^2 - 6a_1a_2 = 0\\
Y^{4}:&\quad -6a_2\mu^2 - 3a_2^2 = 0.
\end{aligned} \tag{4.18}$$

Solving this system yielded the following set of solutions

$$\begin{aligned}
y_0:&\quad \mu = m \in \mathbb{R},\quad a_0 = 0,\quad \frac{1}{3}(c^2-1),\quad a_1 = 0,\quad a_2 = 0,\\
&\quad b_1 = 0,\quad b_2 = 0\\
y_1:&\quad \mu = \pm\frac{1}{2}\sqrt{c^2-1},\quad a_0 = \frac{1}{2}(c^2-1),\quad a_1 = 0,\quad a_2 = \frac{1}{2}(1-c^2),\\
&\quad b_1 = 0,\quad b_2 = 0\\
y_2:&\quad \mu = \pm\frac{1}{2}\sqrt{1-c^2},\quad a_0 = \frac{1}{6}(1-c^2),\quad a_1 = 0,\quad a_2 = \frac{1}{2}(c^2-1),\\
&\quad b_1 = 0,\quad b_2 = 0\\
y_3:&\quad \mu = \pm\frac{1}{2}\sqrt{c^2-1},\quad a_0 = \frac{1}{2}(c^2-1),\quad a_1 = 0,\quad a_2 = 0,\\
&\quad b_1 = 0,\quad b_2 = \frac{1}{2}(1-c^2)\\
y_4:&\quad \mu = \pm\frac{1}{2}\sqrt{1-c^2},\quad a_0 = \frac{1}{6}(1-c^2),\quad a_1 = 0,\quad a_2 = 0,
\end{aligned} \tag{4.19}$$

$$b_1 = 0, \quad b_2 = \frac{1}{2}(c^2 - 1)$$

$$y_5: \quad \mu = \pm\frac{1}{4}\sqrt{c^2-1}, \quad a_0 = \frac{1}{4}(c^2-1), \quad a_1 = 0, \quad a_2 = \frac{1}{8}(1-c^2),$$

$$b_1 = 0, \quad b_2 = \frac{1}{8}(1 - c^2)$$

$$y_6: \quad \mu = \pm\frac{1}{4}\sqrt{1-c^2}, \quad a_0 = \frac{1}{12}(c^2-1), \quad a_1 = 0, \quad a_2 = \frac{1}{8}(c^2-1),$$

$$b_1 = 0, \quad b_2 = \frac{1}{8}(c^2 - 1). \tag{4.19}$$

As before, some solutions are paied due to their similarities, differing only in parity. Substituting these solutions back into the nonlinear pde yields meaningful results.

For $c^2 > 1$ (supercritical case), y_0 yields trivial solutions, while y_1 and y_2 yield the soliton solutions

$$u_1(x,t)_{\text{ext std}} = \frac{c^2-1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) \tag{4.20}$$

$$u_2(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1 - 3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right], \tag{4.21}$$

whereas y_3 and y_4 yielded the non-soliton traveling wave solutions

$$u_3(x,t)_{\text{ext std}} = -\frac{c^2-1}{2}\operatorname{csch}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) \tag{4.22}$$

$$u_4(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1 - 3\coth^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]. \tag{4.23}$$

Although y_5 and y_6 initially appear to produce distinct solutions, they do not represent new 2-soliton solutions. The presence of the two terms and the lack of genuine uniqueness, reveals that they are in fact equivalent to previously established solutions as in

$$u_5(x,t)_{\text{ext std}} = -\frac{c^2-1}{8}\left[\coth^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)\right. \tag{4.24}$$

$$+\tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)-2\Bigg]$$

$$= -\frac{c^2-1}{2}\operatorname{csch}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)$$

$$= u_3(x,t)_{\text{ext std}} \tag{4.24}$$

$$u_6(x,t)_{\text{ext std}} = \frac{c^2-1}{24}\Bigg[3\coth^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)$$

$$+3\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)+2\Bigg]$$

$$= u_4(x,t)_{\text{ext std}}. \tag{4.25}$$

For the opposite regime where $c^2 < 1$ (subcritical case), y_1, y_2, y_3 and y_4 give the plane periodic solutions

$$u_7(x,t)_{\text{ext std}} = \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right) \tag{4.26}$$

$$u_8(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] \tag{4.27}$$

$$u_9(x,t)_{\text{ext std}} = \frac{c^2-1}{2}\csc^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right) \tag{4.28}$$

$$u_{10}(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1+3\cot^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] \tag{4.29}$$

Again, y_5 and y_6 in this regime lead to non-unique solutions because

$$u_{11}(x,t)_{\text{ext std}} = \frac{c^2-1}{8}\left[\cot^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)+\tan^2\left(\frac{\sqrt{1-c^2}}{4}\right)+2\right]$$

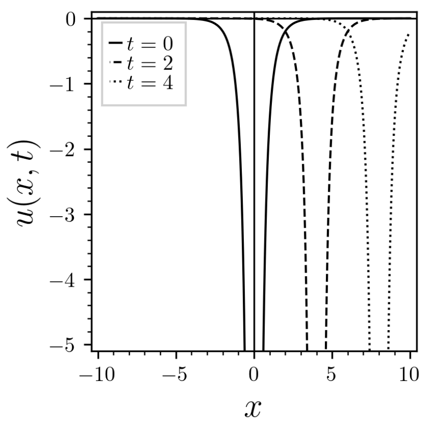
$$= \frac{c^2-1}{2}\csc^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

$$= u_9(x,t)_{\text{ext std}} \tag{4.30}$$

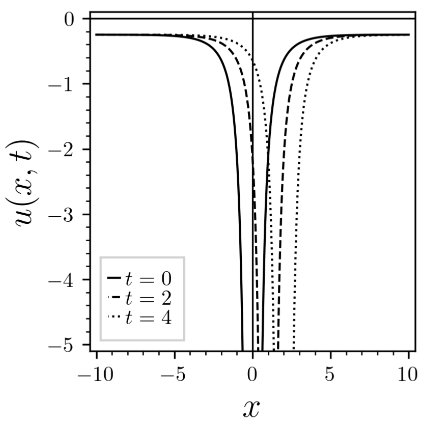
$$u_{12}(x,t)_{\text{ext std}} = -\frac{c^2-1}{24}\Bigg[3\cot^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right) \tag{4.31}$$

$$+3\tan^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)-2\Bigg]$$

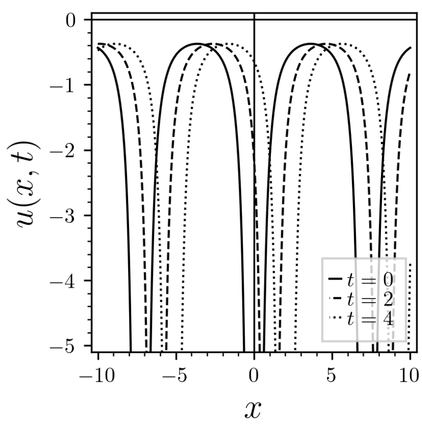
$$= u_{10}(x,t)_{\text{ext std}}. \tag{4.31}$$



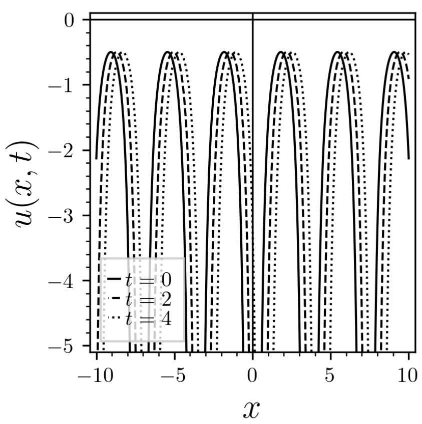
(a): $u_{3,\text{ext std}} : c = 2, |x| \le 10$



(b): $u_{4,\text{ext std}} : c = \frac{1}{2}, |x| \le 10$



(c): $u_{9,\text{ext std}} : c = 2, |x| \le 10$



(d): $u_{10,\text{ext std}} : c = \frac{1}{2}, |x| \le 10$

Figure 5: Plots of the additional solutions to the classical Boussinesq equation via extended standard tanh method, with $t = 0, 2, 4$. The other solutions are found in Figure 4.

The solutions u_3 and u_9 involving csch and csc functions, plotted in Figure 5, often represent waves with singularities or different asymptotic behaviors compared to the sech or sec type solutions. The inclusion of Y^{-k} terms effectively doubles the set of obtainable solution forms. The algebraic manipulations become more involved, and the use of a computer algebra system (cas), SageMath as our particular choice, was noted as beneficial for solving the resulting system of equations for the coefficients.

This step confirmed the power of the extended method in uncovering a richer variety of exact solutions, all of which are consistent with existing literature [19]. It also highlighted that some combinations of coefficients might lead to redundant solutions, as seen with u_5 and u_6.

4.3. Computing the derivatives d_ξ, d_ξ^2

The core of our generalization lies in the novel ansatz

$$Y_{p,\xi} = Y_p(\mu\xi) = (1+p)\frac{\tanh\dfrac{\mu\xi}{2}}{1+p\tanh^2\dfrac{\mu\xi}{2}}, \qquad 0 \le p \le 1, \quad p \in \mathbb{R}. \tag{4.32}$$

introduced as a new independent variable where $\xi = x - ct$ and μ is wave number. To substitute this ansatz into the ode derived from the Boussinesq equation, we expressed its derivatives $\mathrm{d}_\xi Y_p$ and $\mathrm{d}_\xi^2 Y_p$ in terms of Y_p itself. This subsection details this crucial mathematical step.

The derivation involved an auxiliary variable transformation

$$\begin{aligned}
&\tanh\frac{\mu\xi}{2} = \frac{1}{\sqrt{p}}\tanh\frac{\mu\omega}{2} \\
&\implies Y_{p,\xi} = (1+p)\frac{\dfrac{1}{\sqrt{p}}\tanh\dfrac{\mu\omega}{2}}{1+p\dfrac{1}{p}\tanh^2\dfrac{\mu\omega}{2}} \\
&\qquad\quad = \frac{p+1}{2\sqrt{p}}\frac{2\tanh\dfrac{\mu\omega}{2}}{1+\tanh^2\dfrac{\mu\omega}{2}} \\
&\qquad\quad = \frac{p+1}{2\sqrt{p}}\tanh\mu\omega \\
&\qquad\quad = Y_p(\omega) \equiv Y_{p,\omega}.
\end{aligned} \tag{4.33}$$

Note that $\mathrm{d}_\xi = \mathrm{d}_\omega Y_p \cdot \mathrm{d}_{Y_p} = \mathrm{d}_\xi\omega \cdot \mathrm{d}_\omega Y_p \cdot \mathrm{d}_{Y_p}$ and $\omega = \frac{2}{\mu}\operatorname{arctanh}\left(\sqrt{p}\tanh\frac{\mu\xi}{2}\right)$. Applying the chain rule, we first computed

$$\begin{aligned}
\mathrm{d}_\omega Y_p &= \mathrm{d}_\omega Y_{p,\omega} \\
&= \frac{p+1}{2\sqrt{p}}\mu(1-\tanh^2\mu\omega)
\end{aligned} \tag{4.34}$$

$$
\begin{aligned}
&= \frac{p+1}{2\sqrt{p}}\mu\left[1-\left(\frac{2\sqrt{p}}{p+1}\right)^2\left(\frac{p+1}{2\sqrt{p}}\right)^2\tanh^2\mu\omega\right]\\
&= \frac{p+1}{2\sqrt{p}}\mu\left[1-\left(\frac{2\sqrt{p}}{p+1}\right)^2 Y_{p,\omega}^2\right]\\
&= \frac{p+1}{2\sqrt{p}}\left(\frac{2\sqrt{p}}{p+1}\right)^2\mu\left[\left(\frac{p+1}{2\sqrt{p}}\right)^2 - Y_{p,\omega}^2\right]\\
&= \frac{\mu}{q_p}(q_p^2 - Y_{p,\omega}^2)
\end{aligned}
\tag{4.34}
$$

where $q_p \equiv \frac{p+1}{2\sqrt{p}}$. Next, we computed

$$
\begin{aligned}
\mathrm{d}_\xi\omega &= \frac{1}{\sqrt{p}}\frac{p-\tanh^2\frac{\mu\omega}{2}}{1-\tanh^2\frac{\mu\omega}{2}}\\
&= \frac{1}{\sqrt{p}}\frac{p+1}{2}\left[(1-1)+\frac{2p-2\tanh^2\frac{\mu\omega}{2}}{(p+1)\Big(1-\tanh^2\frac{\mu\omega}{2}\Big)}\right]\\
&= \frac{1}{\sqrt{p}}\frac{p+1}{2}\left[1+\frac{-(p-1)\Big(1-\tanh^2\frac{\mu\omega}{2}\Big)+2p-2\tanh^2\frac{\mu\omega}{2}}{(p+1)\Big(1-\tanh^2\frac{\mu\omega}{2}\Big)}\right]\\
&= \frac{1}{\sqrt{p}}\frac{p+1}{2}\left[1+\frac{(p-1)\Big(1+\tanh^2\frac{\mu\omega}{2}\Big)}{(p+1)\Big(1-\tanh^2\frac{\mu\omega}{2}\Big)}\right]\\
&= \frac{1}{\sqrt{p}}\frac{p+1}{2}\left\{1+\frac{p-1}{p+1}\left[\left(\frac{1-\tanh^2\frac{\mu\omega}{2}}{1+\tanh^2\frac{\mu\omega}{2}}\right)^2\right]^{1/2}\right\}.
\end{aligned}
\tag{4.35}
$$

To simplify the innermost term, we have

$$
\begin{aligned}
\left(\frac{1-\tanh^2\frac{\mu\omega}{2}}{1+\tanh^2\frac{\mu\omega}{2}}\right)^2 &= \frac{1-2\tanh^2\frac{\mu\omega}{2}+\tanh^4\frac{\mu\omega}{2}}{\Big(1+\tanh^2\frac{\mu\omega}{2}\Big)^2}\\
&= \frac{1+2\tanh^2\frac{\mu\omega}{2}+\tanh^4\frac{\mu\omega}{2}-4\tanh^2\frac{\mu\omega}{2}}{\Big(1+\tanh^2\frac{\mu\omega}{2}\Big)^2}
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
&= \frac{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2 - 4\tanh^2\frac{\mu\omega}{2}}{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2} \\
&= 1 - \frac{4\tanh^2\frac{\mu\omega}{2}}{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2} \\
&= 1 - \tanh^2\mu\omega \\
&= \left(\frac{2\sqrt{p}}{p+1}\right)^2\left[\left(\frac{p+1}{2\sqrt{p}}\right)^2 - \left(\frac{p+1}{2\sqrt{p}}\right)^2\tanh^2\mu\omega\right] \\
&= \frac{1}{q_p^2}\left(q_p^2 - Y_{p,\xi}^2\right). \qquad (4.36)
\end{aligned}
$$

With $r_p \equiv \frac{p-1}{2\sqrt{p}} = \frac{p-1}{p+1}q_p$, we obtained

$$
\begin{aligned}
\mathrm{d}_\xi\omega &= q_p\left[1 + \frac{p-1}{p+1}q_p\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right] \\
&= q_p\left[1 + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right]. \qquad (4.37)
\end{aligned}
$$

Finally, the resulting expressions for the first and second derivatives were

$$
\begin{aligned}
\mathrm{d}_\xi &= \mathrm{d}_\xi\omega \cdot \mathrm{d}_\omega Y_p \cdot d_{Y_p} \\
&= q_p\left[1 + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right]\frac{\mu}{q_p}\left(q_p^2 - Y_{p,\xi}^2\right)\mathrm{d}_{Y_p} \\
&= \mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]\mathrm{d}_{Y_p} \qquad (4.38)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}_\xi^2 &= \mathrm{d}_\xi\left\{\mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]\mathrm{d}_{Y_p}\right\} \\
&= \mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right] \\
&\qquad \mathrm{d}_{Y_p}\left\{\mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]\mathrm{d}_{Y_p}\right\} \\
&= \mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right] \\
&\qquad \left\{\mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]\mathrm{d}_{Y_p}^2\right. \qquad (4.39)
\end{aligned}
$$

$$+\mu\left[(0-2Y_{p,\xi})+r_p\frac{1}{2}(0-2Y_{p,\xi})\left(q_p^2-Y_{p,\xi}^2\right)^{-1/2}\right]\mathrm{d}_{Y_p}\Bigg\}$$

$$\begin{aligned}
&=\mu^2\left[\left(q_p^2-Y_{p,\xi}^2\right)+r_p\left(q_p^2-Y_{p,\xi}^2\right)^{1/2}\right]^2\mathrm{d}_{Y_p}^2\\
&\quad+\mu^2\left[\left(q_p^2-Y_{p,\xi}^2\right)+r_p\left(q_p^2-Y_{p,\xi}^2\right)^{1/2}\right]\\
&\qquad\left[-2Y_{p,\xi}-r_pY_{p,\xi}\left(q_p^2-Y_{p,\xi}^2\right)^{-1/2}\right]\mathrm{d}_{Y_p}.
\end{aligned}\tag{4.39}$$

This computation provides an improvement in conciseness over previous the operational rules for how derivatives of $u(x,t)$, expressed as a series in Y_p, transform. It is also an improvement in conciseness compared to previous work [5]. The complexity of these derivative operators, particularly the appearance of terms like $\left(q_p^2-Y_p^2\right)^{\frac{1}{2}}$, highlights the algebraic intricacy of the generalized method. This explains the necessity of introducing a forcing function $F\left(Y_p\right)$ when $p\neq 1$. Accurate derivative forms are essential for the subsequent application of the generalized method.

4.4. Solutions via generalized tanh method

Having explored the fundamental solutions of the classical Boussinesq equation using the standard and extended standard tanh methods, and with the derivative operators now established, we proceeded to apply the generalized tanh method.

The classical Boussinesq equation

$$\partial_t^2u-\partial_x^2u-\partial_x^2(3u^2)-\partial_x^4u=0\tag{4.40}$$

was transformed into a nonlinear ode via the transformation $\xi=\mu(x-ct)$ as detailed in Figure 3 which yielded

$$c^2\mathrm{d}_\xi^2u-\mathrm{d}_\xi^2u-3\mathrm{d}_\xi^2u^2-\mathrm{d}_\xi^4u=0\tag{4.41}$$

where c and μ represent wave speed and wave number, respectively. We integrated this ode wrt ξ twice as in

$$\begin{aligned}
&(c^2-1)\mathrm{d}_\xi^2u-3\mathrm{d}_\xi^2u^2-\mathrm{d}_\xi^4u=0\\
\implies &(c^2-1)\mathrm{d}_\xi u-3\mathrm{d}_\xi u^2-\mathrm{d}_\xi^3u=0
\end{aligned}\tag{4.42}$$

$$\implies (c^2-1)u - 3u^2 - \mathrm{d}_\xi^2 u = 0. \tag{4.42}$$

Balancing the highest-order nonlinear term with highest-order derivative as per the previously discussed mapping, we got

$$\begin{aligned} u^2 &= \mathrm{d}_\xi^2 u \\ \implies 2M &= M+2 \\ M &= 2. \end{aligned} \tag{4.43}$$

This method admits the use of the series substitution

$$\begin{aligned} u(x,t) = S(Y) &= \sum_{i=0}^{M} a_i Y_i \\ \implies U(x,t) &= a_0 + a_1 Y + a_2 Y^2 \end{aligned} \tag{4.44}$$

as solution to the ode, and consequently the pde, based on the new independent variable

$$Y_{p,\xi} = Y_p(\mu\xi) = (1+p)\frac{\tanh\frac{\mu\xi}{2}}{1+p\tanh^2\frac{\mu\xi}{2}}, \quad 0 \le p \le 1, \quad p \in \mathbb{R}. \tag{4.45}$$

as our ansatz. Recall the previously computed derivatives

$$\begin{aligned} \mathrm{d}_\xi &= \mathrm{d}_\xi\omega \cdot \mathrm{d}_\omega Y_p \cdot d_{Y_p} \\ &= q_p\left[1 + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right]\frac{\mu}{q_p}\left(q_p^2 - Y_{p,\xi}^2\right)\mathrm{d}_{Y_p} \\ &= \mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]\mathrm{d}_{Y_p} \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} \mathrm{d}_\xi^2 &= \mathrm{d}_\xi \cdot \left(\mathrm{d}_\xi Y \cdot \mathrm{d}_Y\right) \\ &= \mu^2\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]^2 \mathrm{d}_{Y_p}^2 \\ &\quad + \mu^2\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right] \\ &\qquad \left[-2Y_{p,\xi} - r_p Y_{p,\xi}\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right]\mathrm{d}_{Y_p}. \end{aligned} \tag{4.47}$$

We substituted the series and the derivative expressions into the ode, and grouped the terms by powers of Y, which yielded

$$
\begin{aligned}
0 = &(c^2-1)(a_0+a_1Y+a_2Y^2) \\
&-3(a_0+a_1Y+a_2Y^2)^2 \\
&+\Bigg[\mu^2\Big(\big(q_p^2-Y_{p,\xi}^2\big)+r_p\big(q_p^2-Y_{p,\xi}^2\big)^{1/2}\Big)^2\cdot \mathrm{d}_{Y_p}^2 \\
&\quad+\mu^2\Big(\big(q_p^2-Y_{p,\xi}^2\big)+r_p\big(q_p^2-Y_{p,\xi}^2\big)^{1/2}\Big) \\
&\qquad\Big(-2Y_{p,\xi}-r_pY_{p,\xi}\big(q_p^2-Y_{p,\xi}^2\big)^{-1/2}\Big)\cdot \mathrm{d}_{Y_p}\Bigg] \\
&\qquad(a_0+a_1Y+a_2Y^2)
\end{aligned} \tag{4.48}
$$

Equating coefficients of each power of Y to 0 resulted in the system of equations

$$
\begin{aligned}
Y_p^0: &\quad -2a_2\mu^2q^4-2a_2\mu^2q^2r^2+a_0c^2-3a_0^2-a_0=0 \\
Y_p^1: &\quad 2a_1\mu^2q^2+a_1\mu^2r^2+a_1c^2-6a_0a_1-a_1=0 \\
Y_p^2: &\quad 8a_2\mu^2q^2+4a_2\mu^2r^2+a_2c^2-3a_1^2-6a_0a_2-a_2=0 \\
Y_p^3: &\quad -2a_1\mu^2-6a_1a_2=0 \\
Y_p^4: &\quad -6a_2\mu^2-3a_2^2=0 \\
Y_p^2\big(q_p^2-Y_p^2\big)^{-1/2} &: 2a_2\mu^2q^2r=0 \\
Y_p^3\big(q_p^2-Y_p^2\big)^{-1/2} &: a_1\mu^2r=0 \\
Y_p^4\big(q_p^2-Y_p^2\big)^{-1/2} &: -2a_2\mu^2r=0 \\
Y_p^0\big(q_p^2-Y_p^2\big)^{1/2} &: -4a_2\mu^2q^2r=0 \\
Y_p^1\big(q_p^2-Y_p^2\big)^{1/2} &: 2a_1\mu^2r=0 \\
Y_p^2\big(q_p^2-Y_p^2\big)^{1/2} &: a_1\mu^2q^2r=0.
\end{aligned} \tag{4.49}
$$

Importantly, terms involving non-integer powers of Y_p, specifically those with $\big(q_p^2-Y_p^2\big)^{\pm\frac{1}{2}}$, emerged. These terms cannot be balanced by integer powers of Y_p alone. To address this, we introduce a forcing function $F\big(Y_p\big)$ to isolate the special terms

$$
F(Y)=2a_2\mu^2q^2rY_p^2\big(q_p^2-Y_p^2\big)^{-1/2}+a_1\mu^2rY_p^3\big(q_p^2-Y_p^2\big)^{-1/2} \tag{4.50}
$$

$$
\begin{aligned}
&-2a_2\mu^2 rY_p^4\big(q_p^2 - Y_p^2\big)^{-1/2} - 4a_2\mu^2 q^2 r\big(q_p^2 - Y_p^2\big)^{1/2}\\
&+2a_1\mu^2 rY_p\big(q_p^2 - Y_p^2\big)^{1/2} + a_1\mu^2 q^2 rY_p^2\big(q_p^2 - Y_p^2\big)^{1/2}\\
=&\ \big[2a_2\mu^2 q^2 rY_p^2 + a_1\mu^2 rY_p^3 - 2a_2\mu^2 rY_p^4\big]\big(q_p^2 - Y_p^2\big)^{-1/2}\\
&+\big[-4a_2\mu^2 q^2 r + 2a_1\mu^2 rY_p + a_1\mu^2 q^2 rY_p^2\big]\big(q_p^2 - Y_p^2\big)^{1/2} \qquad (4.50)
\end{aligned}
$$

which we note can be further simplified.

By equating the original nonlinear ode, and consequently the nonlinear pde, to this forcing function, we obtain a forced version of the Boussinesq equation

$$
\begin{aligned}
&(c^2-1)u - 3u^2 - \mathrm{d}_\xi^2 u = F(Y)\\
&\Longrightarrow \partial_t^2 u - \partial_x^2 u - \partial_x^2(3u^2) - \partial_x^4 u = F(Y). \qquad (4.51)
\end{aligned}
$$

This modification allowed us to eliminate terms with non-integral powers of Y. Importantly, the original, unforced Boussinesq equation is recovered by setting $p = 1$, which makes $r_p = 0$ and therefore $F\big(Y_p\big) = 0$.

Setting the forcing function aside, the remaining terms with integer powers of Y yield this reduced system of equations

$$
\begin{aligned}
Y_p^0:&\quad -2a_2\mu^2 q^4 - 2a_2\mu^2 q^2 r^2 + a_0 c^2 - 3a_0^2 - a_0 = 0\\
Y_p^1:&\quad 2a_1\mu^2 q^2 + a_1\mu^2 r^2 + a_1 c^2 - 6a_0 a_1 - a_1 = 0\\
Y_p^2:&\quad 8a_2\mu^2 q^2 + 4a_2\mu^2 r^2 + a_2 c^2 - 3a_1^2 - 6a_0 a_2 - a_2 = 0\\
Y_p^3:&\quad -2a_1\mu^2 - 6a_1 a_2 = 0\\
Y_p^4:&\quad -6a_2\mu^2 - 3a_2^2 = 0. \qquad (4.52)
\end{aligned}
$$

Using SageMath as our cas, and substituting the definitions

$$
\begin{aligned}
q_p &\equiv \frac{p+1}{2\sqrt{p}}\\
r_p &\equiv \frac{p-1}{2\sqrt{p}} = \frac{p-1}{p+1}q_p, \qquad (4.53)
\end{aligned}
$$

we obtained the following sets of solutions

$$
y_0:\quad a_0 = 0,\quad \frac{1}{3}(c^2-1),\quad a_1 = 0,\quad a_2 = 0,\quad \mu = m \in \mathbb{R}; \qquad (4.54)
$$

$$
\begin{aligned}
y_1: \quad a_0 &= \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right), \quad a_1 = 0,\\
a_2 &= (1-c^2)\frac{2p}{\sqrt{(3p^2+1)(p^2+3)}},\\
\mu &= \pm\sqrt{c^2-1}\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}};\\
y_2: \quad a_0 &= \frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right), \quad a_1 = 0,\\
a_2 &= (c^2-1)\frac{2p}{\sqrt{(3p^2+1)(p^2+3)}},\\
\mu &= \pm\sqrt{1-c^2}\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}. \qquad (4.54)
\end{aligned}
$$

An aside: without pre-substituting q and r, the sets of solutions originally was

$$
\begin{aligned}
y_0: \quad a_0 &= 0, \quad \frac{1}{3}(c^2-1), \quad a_1 = 0, \quad a_2 = 0, \quad \mu = m \in \mathbb{R};\\
y_1: \quad a_0 &= \frac{c^2-1}{6}\left(1+\frac{2q^2+r^2}{\sqrt{q^4+q^2r^2+r^4}}\right), \quad a_1 = 0,\\
a_2 &= \frac{1-c^2}{2}\frac{1}{\sqrt{q^4+q^2r^2+r^4}},\\
\mu &= \pm\frac{\sqrt{c^2-1}}{2}\frac{1}{\sqrt[4]{q^4+q^2r^2+r^4}};\\
y_2: \quad a_0 &= \frac{c^2-1}{6}\left(1-\frac{2q^2+r^2}{\sqrt{q^4+q^2r^2+r^4}}\right), \quad a_1 = 0,\\
a_2 &= \frac{c^2-1}{2}\frac{1}{\sqrt{q^4+q^2r^2+r^4}},\\
\mu &= \pm\frac{\sqrt{1-c^2}}{2}\frac{1}{\sqrt[4]{q^4+q^2r^2+r^4}}. \qquad (4.55)
\end{aligned}
$$

We note that some solutions are paired due to similarities, differing only in parity. Substituting these solutions back into the forced nonlienar pde gave us familiar results.

Specifically, when $c^2 > 1$, y_0 gives trivial solutions, while y_1 and y_2 provide the soliton solutions

$$\begin{aligned}
u_1(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}&\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)\\
&+(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}\\
&\left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
u_2(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}&\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)\\
&+(c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}\\
&\left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2 .
\end{aligned} \tag{4.57}$$

In the opposite regime where $c^2 < 1$, y_1 and y_2 yield the plane periodic solutions

$$\begin{aligned}
u_3(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}&\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)\\
&+(c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}\\
&\left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2
\end{aligned} \tag{4.58}$$

$$\begin{aligned}
u_4(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}&\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)\\
&+(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}\\
&\left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2
\end{aligned} \tag{4.59}$$

As a crucial verification, setting $p = 1$ in these generalized solutions produces the unforced particular solutions obtained via the standard tanh method as in

$$\begin{aligned}
u_1(x,t,p=1)_{\text{gen}} &= \frac{c^2-1}{2} + 2(1-c^2)\left[\frac{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2 \\
&= \frac{c^2-1}{2}\left[1-\tanh^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] \\
&= \frac{c^2-1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) \\
&= u_1(x,t)_{\text{ext std}} = u_1(x,t)_{\text{std}}
\end{aligned} \tag{4.60}$$

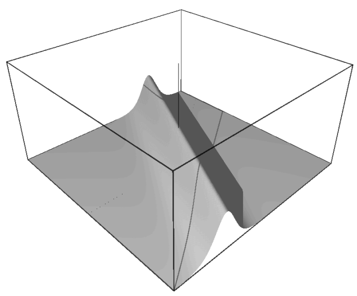
$$\begin{aligned}
u_2(x,t,p=1)_{\text{gen}} &= -\frac{c^2-1}{6} + 2(c^2-1)\left[\frac{\tanh\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2 \\
&= -\frac{c^2-1}{6}\left[1-3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right] \\
&= u_2(x,t)_{\text{ext std}} = u_2(x,t)_{\text{std}}
\end{aligned} \tag{4.61}$$

$$\begin{aligned}
u_3(x,t,p=1)_{\text{gen}} &= \frac{c^2-1}{2} + 2(c^2-1)\left[\frac{\tan\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2 \\
&= \frac{c^2-1}{2}\left[1+\tan^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right] \\
&= \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right) \\
&= u_7(x,t)_{\text{ext std}} = u_3(x,t)_{\text{std}}
\end{aligned} \tag{4.62}$$

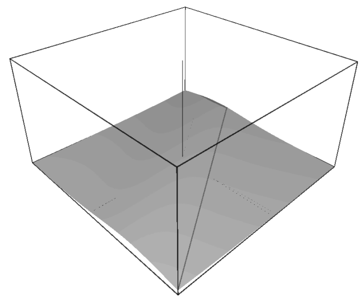
$$\begin{aligned}
u_4(x,t,p=1)_{\text{gen}} &= -\frac{c^2-1}{6} + 2(1-c^2)\left[\frac{\tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2 \\
&= -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] \\
&= u_8(x,t)_{\text{ext std}} = u_4(x,t)_{\text{std}}
\end{aligned} \tag{4.63}$$

In summary, the generalized tanh method, for $p \neq 1$, yields solutions to the forced Boussinesq equation due to the emergence of terms involving non-integer powers of Y_p, represented by the forcing function $F(Y_p)$. Setting $p = 1$ eliminates $F(Y_p)$, resulting in solutions to the original, unforced Boussinesq equation.

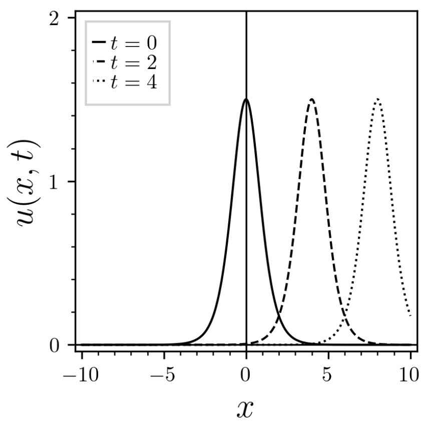
This approach introduces a valuable parameter p, which provides a continuous deformation of the standard solutions while revealing solutions to related forced systems. See plots in Figure 6 and Figure 7 for the solutions.



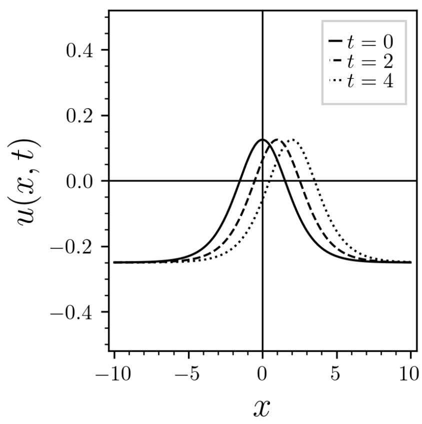
(a): $u_{1,\text{gen}} : c = 2, |x, t| \leq 4$



(b): $u_{2,\text{gen}} : c = \frac{1}{2}, |x, t| \leq 4$

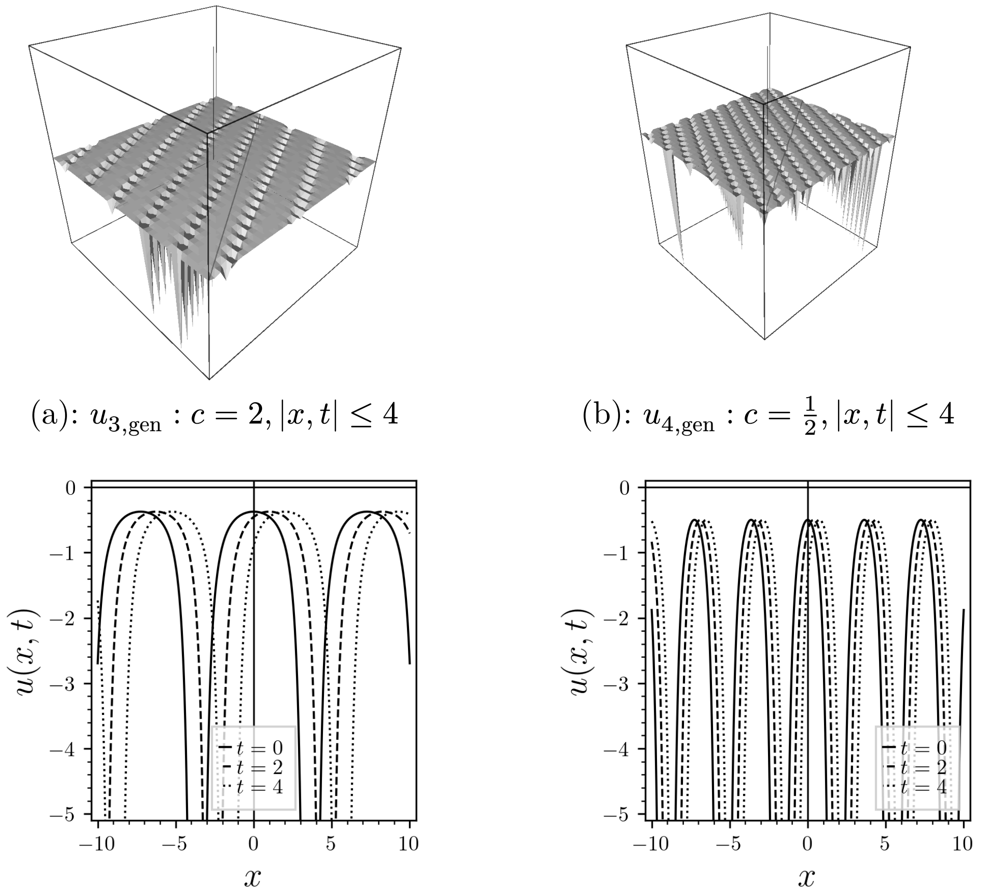


(c): $u_{1,\text{gen}} : c = 2, |x| \leq 10$



(d): $u_{2,\text{gen}} : c = \frac{1}{2}, |x| \leq 10$

Figure 6: Plots of the soliton solutions to the classical Boussinesq equation via generalized tanh method, with $t = 0, 2, 4$.



(c): $u_{3,\text{gen}} : c = 2, |x| \leq 10$ (d): $u_{4,\text{gen}} : c = \frac{1}{2}, |x| \leq 10$

Figure 7: Plots of the plane periodic solutions to the classical Boussinesq equation via generalized tanh method, with $t = 0, 2, 4$.

4.5. Solutions via extended generalized tanh method

This subsection integrates the strengths of both the series extension and the generalization introduced via the ansatz Y_p. Similar to our treatment of the extended standard tanh method, the extended generalized tanh method largely followed same procedures, with the exception that we now employed a more comprehensive series expansion previously discussed inthat incorporated negative powers of Y_p as in

$$\begin{aligned} 0 = &(c^2 - 1)(a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}) \\ &-3(a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2})^2 \end{aligned} \tag{4.64}$$

$$
+\Bigg[\mu^2\Big(\big(q_p^2 - Y_{p,\xi}^2\big) + r_p\big(q_p^2 - Y_{p,\xi}^2\big)^{1/2}\Big)^2 \cdot \mathrm{d}_{Y_p}^2
$$

$$
+\mu^2\Big(\big(q_p^2 - Y_{p,\xi}^2\big) + r_p\big(q_p^2 - Y_{p,\xi}^2\big)^{1/2}\Big)
$$

$$
\Big(-2Y_{p,\xi} - r_p Y_{p,\xi}\big(q_p^2 - Y_{p,\xi}^2\big)^{-1/2}\Big) \cdot \mathrm{d}_{Y_p}\Bigg]
$$

$$
(a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}). \tag{4.64}
$$

This led to the following, more extensive system of equations

$$
\begin{aligned}
Y^{-4}: &\quad -6b_2\mu^2 q^4 - 6b_2\mu^2 q^2 r^2 - 3b_2^2 = 0 \\
Y^{-3}: &\quad -2b_1\mu^2 q^4 - 2b_1\mu^2 q^2 r^2 - 6b_1 b_2 = 0 \\
Y^{-2}: &\quad 8b_2\mu^2 q^2 + 4b_2\mu^2 r^2 + b_2 c^2 - 3b_1^2 - 6a_0 b_2 - b_2 = 0 \\
Y^{-1}: &\quad 2b_1\mu^2 q^2 + b_1\mu^2 r^2 + b_1 c^2 - 6a_0 b_1 - 6a_1 b_2 - b_1 = 0 \\
Y^{0}: &\quad -2a_2\mu^2 q^4 - 2a_2\mu^2 q^2 r^2 + a_0 c^2 - 2b_2\mu^2 - 3a_0^2 - 6a_1 b_1 \\
&\qquad -6a_2 b_2 - a_0 = 0 \\
Y^{1}: &\quad 2a_1\mu^2 q^2 + a_1\mu^2 r^2 + a_1 c^2 - 6a_0 a_1 - 6a_2 b_1 - a_1 = 0 \\
Y^{2}: &\quad 8a_2\mu^2 q^2 + 4a_2\mu^2 r^2 + a_2 c^2 - 3a_1^2 - 6a_0 a_2 - a_2 = 0 \\
Y^{3}: &\quad -2a_1\mu^2 - 6a_1 a_2 = 0 \\
Y^{4}: &\quad -6a_2\mu^2 - 3a_2^2 = 0
\end{aligned} \tag{4.65}
$$

with the terms involving non-integer powers of Y_p already separated

$$
\begin{aligned}
Y_p^{-2}\big(q_p^2 - Y_p^2\big)^{-1/2} &: -2b_2\mu^2 q^2 r = 0 \\
Y_p^{-1}\big(q_p^2 - Y_p^2\big)^{-1/2} &: -b_1\mu^2 q^2 r = 0 \\
Y_p^{0}\big(q_p^2 - Y_p^2\big)^{-1/2} &: 2b_2\mu^2 r = 0 \\
Y_p^{1}\big(q_p^2 - Y_p^2\big)^{-1/2} &: a_1\mu^2 q^2 r + b_1\mu^2 r = 0 \\
Y_p^{2}\big(q_p^2 - Y_p^2\big)^{-1/2} &: 2a_2\mu^2 q^2 r = 0 \\
Y_p^{3}\big(q_p^2 - Y_p^2\big)^{-1/2} &: -a_1\mu^2 r = 0 \\
Y_p^{4}\big(q_p^2 - Y_p^2\big)^{-1/2} &: -2a_2\mu^2 r = 0 \\
Y_p^{-4}\big(q_p^2 - Y_p^2\big)^{1/2} &: -12b_2\mu^2 q^2 r = 0
\end{aligned} \tag{4.66}
$$

$$
\begin{aligned}
Y_p^{-3}\left(q_p^2-Y_p^2\right)^{1/2} &: -4b_1\mu^2q^2r=0\\
Y_p^{-2}\left(q_p^2-Y_p^2\right)^{1/2} &: 8b_2\mu^2r=0\\
Y_p^{-1}\left(q_p^2-Y_p^2\right)^{1/2} &: 2b_1\mu^2r=0\\
Y_p^{0}\left(q_p^2-Y_p^2\right)^{1/2} &: -4a_2\mu^2q^2r-b_1\mu^2q^2r=0\\
Y_p^{1}\left(q_p^2-Y_p^2\right)^{1/2} &: 2a_1\mu^2r=0\\
Y_p^{2}\left(q_p^2-Y_p^2\right)^{1/2} &: 8a_2\mu^2r=0
\end{aligned}
\tag{4.66}
$$

into the forcing function

$$
\begin{aligned}
F(Y) = &-2b_2\mu^2q^2rY_p^{-2}\left(q_p^2-Y_p^2\right)^{-1/2}-b_1\mu^2q^2rY_p^{-1}\left(q_p^2-Y_p^2\right)^{-1/2}\\
&+2b_2\mu^2r\left(q_p^2-Y_p^2\right)^{-1/2}+(a_1\mu^2q^2r+b_1\mu^2r)Y_p\left(q_p^2-Y_p^2\right)^{-1/2}\\
&+2a_2\mu^2q^2rY_p^2\left(q_p^2-Y_p^2\right)^{-1/2}-a_1\mu^2rY_p^3\left(q_p^2-Y_p^2\right)^{-1/2}\\
&-2a_2\mu^2rY_p^4\left(q_p^2-Y_p^2\right)^{-1/2}-12b_2\mu^2q^2rY_p^{-4}\left(q_p^2-Y_p^2\right)^{1/2}\\
&-4b_1\mu^2q^2rY_p^{-3}\left(q_p^2-Y_p^2\right)^{1/2}+8b_2\mu^2rY_p^{-2}\left(q_p^2-Y_p^2\right)^{1/2}\\
&+2b_1\mu^2rY_p^{-1}\left(q_p^2-Y_p^2\right)^{1/2}+(-4a_2\mu^2q^2r-b_1\mu^2q^2r)\left(q_p^2-Y_p^2\right)^{1/2}\\
&+2a_1\mu^2rY_p\left(q_p^2-Y_p^2\right)^{1/2}+8a_2\mu^2rY_p^2\left(q_p^2-Y_p^2\right)^{1/2}\\
= &\big[-2b_2\mu^2q^2rY_p^{-2}-b_1\mu^2q^2rY_p^{-1}+2b_2\mu^2r+(a_1\mu^2q^2r+b_1\mu^2r)Y_p\\
&+2a_2\mu^2q^2rY_p^2-a_1\mu^2rY_p^3-2a_2\mu^2rY_p^4\big]\left(q_p^2-Y_p^2\right)^{-1/2}\\
&+\big[-12b_2\mu^2q^2rY_p^{-4}-4b_1\mu^2q^2rY_p^{-3}+8b_2\mu^2rY_p^{-2}+2b_1\mu^2rY_p^{-1}\\
&+(-4a_2\mu^2q^2r-b_1\mu^2q^2r)+2a_1\mu^2rY_p+8a_2\mu^2rY_p^2\big]\\
&\left(q_p^2-Y_p^2\right)^{1/2}.
\end{aligned}
\tag{4.67}
$$

By equating our nonlinear ode, and therefore our nonlinear pde, to $F\left(Y_p\right)$, the equation becomes forced. In particular, by setting $p=1$, we may recover the unforced system. Then the reduced system of equations has solutions

$$
\begin{aligned}
y_0:\quad & a_0=0,\frac{1}{3}(c^2-1),\quad a_1=0,\quad a_2=0,\quad b_1=0,\quad b_2=0,\\
&\mu=m\in\mathbb{R};
\end{aligned}
\tag{4.68}
$$

$$y_1: \quad a_0 = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right), \quad a_1 = 0,$$
$$a_2 = (1-c^2)\frac{2p}{\sqrt{(3p^2+1)(p^2+3)}}, \quad b_1 = 0, \quad b_2 = 0,$$
$$\mu = \pm\sqrt{c^2-1}\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}};$$
$$y_2: \quad a_0 = \frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right), \quad a_1 = 0,$$
$$a_2 = (c^2-1)\frac{2p}{\sqrt{(3p^2+1)(p^2+3)}}, \quad b_1 = 0, \quad b_2 = 0,$$
$$\mu = \pm\sqrt{1-c^2}\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}};$$
$$y_3: \quad a_0 = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right), \quad a_1 = 0,$$
$$a_2 = 0, \quad b_1 = 0, \quad b_2 = (1-c^2)\frac{(p^2+1)(p+1)^2}{4p\sqrt{(3p^2+1)(p^2+3)}},$$
$$\mu = \pm\sqrt{c^2-1}\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}};$$
$$y_4: \quad a_0 = \frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right), \quad a_1 = 0,$$
$$a_2 = 0, \quad b_1 = 0, \quad b_2 = (c^2-1)\frac{(p^2+1)(p+1)^2}{4p\sqrt{(3p^2+1)(p^2+3)}},$$
$$\mu = \pm\sqrt{1-c^2}\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}; \tag{4.68}$$

Substituting these solutions back into the pde, we obtained familiar results.

In particular, when $c^2 > 1$, y_0 provides trivial solutions, while y_1 and y_2 give the soliton solutions

$$\begin{aligned} u_1(x,t,p)_{\text{ext gen}} = {} & \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right) \\ & +(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}} \end{aligned} \tag{4.69}$$

$$\left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2 \tag{4.69}$$

$$\begin{aligned} u_2(x,t,p)_{\text{ext gen}} = &\frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right) \\ &+(c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}} \\ &\left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2 \end{aligned} \tag{4.70}$$

whereas y_3 and y_4 give the non-soliton traveling wave solutions

$$\begin{aligned} u_3(x,t,p)_{\text{ext gen}} = &\frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right) \\ &+(1-c^2)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}} \\ &\left[\frac{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2 \end{aligned} \tag{4.71}$$

$$\begin{aligned} u_4(x,t,p)_{\text{ext gen}} = &\frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right) \\ &+(c^2-1)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}} \\ &\left[\frac{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2 \end{aligned} \tag{4.72}$$

In the opposite regime where $c^2 < 1$, y_1, y_2, y_3 and y_4 give the plane periodic solutions

$$\begin{aligned} u_5(x,t,p)_{\text{ext gen}} = &\frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right) \\ &+(c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}} \end{aligned} \tag{4.73}$$

$$\left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2 \quad (4.73)$$

$$u_6(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$
$$+(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}$$
$$\left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2 \quad (4.74)$$

$$u_7(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$
$$+(c^2-1)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}}$$
$$\left[\frac{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2 \quad (4.75)$$

$$u_8(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$
$$+(1-c^2)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}}$$
$$\left[\frac{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2 \quad (4.76)$$

As with the generalized tanh method, a crucial verification is performed by setting $p = 1$. This should reduce the forced generalized extended solutions to the unforced extended standard solutions

$$u_1(x,t,p=1)_{\text{ext gen}} = \frac{c^2-1}{2} + 2(1-c^2)\left[\frac{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2 \quad (4.77)$$

$$
\begin{aligned}
&= \frac{c^2-1}{2}\left[1-\tanh^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] \\
&= \frac{c^2-1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) \\
&= u_1(x,t)_{\text{gen}} = u_1(x,t)_{\text{ext std}} = u_1(x,t)_{\text{std}}
\end{aligned} \tag{4.77}
$$

$$
\begin{aligned}
u_2(x,t,p=1)_{\text{ext gen}} &= -\frac{c^2-1}{6} + 2(c^2-1)\left[\frac{\tanh\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2 \\
&= -\frac{c^2-1}{6}\left[1-3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right] \\
&= u_2(x,t)_{\text{gen}} = u_2(x,t)_{\text{ext std}} = u_2(x,t)_{\text{std}}
\end{aligned} \tag{4.78}
$$

$$
\begin{aligned}
u_3(x,t,p=1)_{\text{ext gen}} &= \frac{c^2-1}{2} + \frac{1-c^2}{8}\left[\frac{1+\tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2 \\
&= \frac{c^2-1}{2}\left[1-\coth^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] \\
&= -\frac{c^2-1}{2}\operatorname{csch}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) \\
&= u_3(x,t)_{\text{ext std}}
\end{aligned} \tag{4.79}
$$

$$
\begin{aligned}
u_4(x,t,p=1)_{\text{ext gen}} &= -\frac{c^2-1}{6} + \frac{c^2-1}{8}\left[\frac{1+\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{\tanh\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2 \\
&= -\frac{c^2-1}{6}\left[1-3\coth^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right] \\
&= u_4(x,t)_{\text{ext std}}
\end{aligned} \tag{4.80}
$$

$$
\begin{aligned}
u_5(x,t,p=1)_{\text{ext gen}} &= \frac{c^2-1}{2} + 2(c^2-1)\left[\frac{\tan\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2 \\
&= \frac{c^2-1}{2}\left[1+\tan^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]
\end{aligned} \tag{4.81}
$$

$$= \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

$$= u_3(x,t)_{\text{gen}} = u_7(x,t)_{\text{ext std}} = u_3(x,t)_{\text{std}} \tag{4.81}$$

$$u_6(x,t,p=1)_{\text{ext gen}} = -\frac{c^2-1}{6} + 2(1-c^2)\left[\frac{\tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

$$= -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

$$= u_4(x,t)_{\text{gen}} = u_8(x,t)_{\text{ext std}} = u_4(x,t)_{\text{std}} \tag{4.82}$$

$$u_7(x,t,p=1)_{\text{ext gen}} = \frac{c^2-1}{2} + \frac{c^2-1}{8}\left[\frac{1-\tan^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{\tan\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2$$

$$= \frac{c^2-1}{2}\left[1+\cot^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]$$

$$= \frac{c^2-1}{2}\csc^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

$$= u_9(x,t)_{\text{ext std}} \tag{4.83}$$

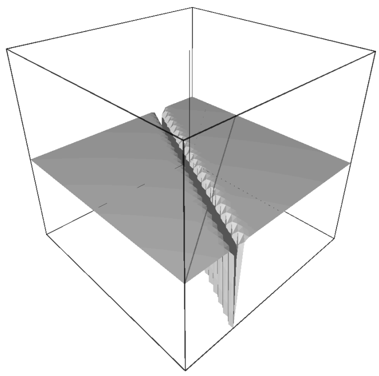
$$u_8(x,t,p=1)_{\text{ext gen}} = -\frac{c^2-1}{6} + \frac{1-c^2}{8}\left[\frac{1-\tan^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{\tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

$$= -\frac{c^2-1}{6}\left[1+3\cot^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

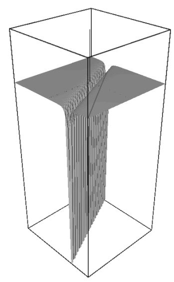
$$= u_{10}(x,t)_{\text{ext std}}. \tag{4.84}$$

Consistent with the generalized tanh method, setting $p = 1$ correctly reduces these solutions to those obtained from the extended standard tanh method. This highlights that the extended generalized method truly encompasses previous methods while providing additional tunable solution families. The 8 unique families of solutions (for $p = 1$, with potential for different families for other values of p) represent a significant expansion of known exact solutions for the forced Boussinesq equation. The ability to tune these solutions via p is a key contribution. See plots in Figure 8 and Figure 9 for the solutions.

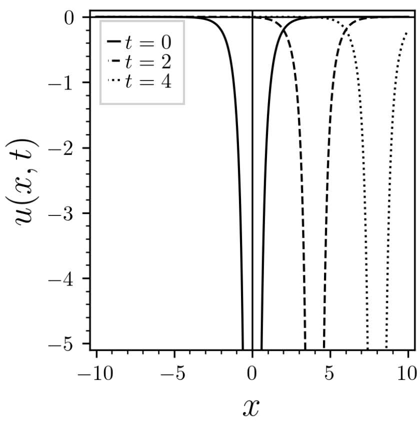
In conclusion, by extending the generalized tanh method, 14 sets of solutions that satisfy the system of equations were identified. Two of these solutions are trivial. Additionally, two solutions, initially appearing as two-term sums of hyperbolic functions, correspond to existing solutions and are not unique. Thus, 8 families of solutions were obtained: two solitons, two non-soliton traveling waves, and four plane periodic solutions. For $p \neq 1$, we are solving a forced version of the equation.



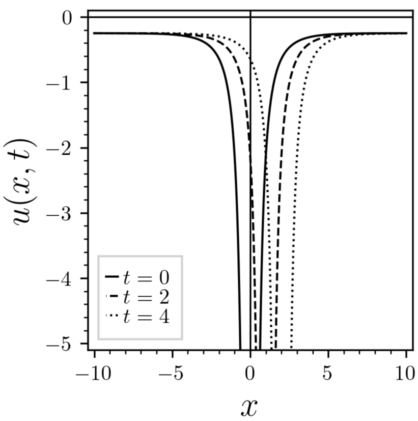
(a): $u_{3,\text{ext gen}} : c = 2, |x, t| \leq 4$



(b): $u_{4,\text{ext gen}} : c = \frac{1}{2}, |x, t| \leq 4$



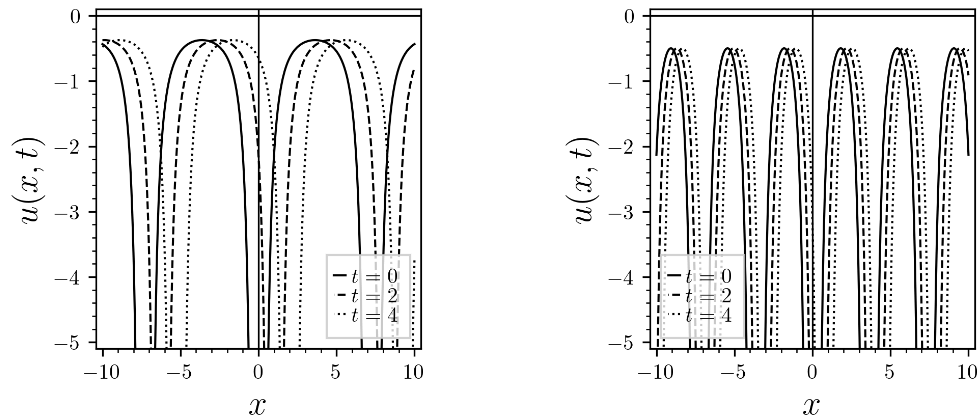
(c): $u_{3,\text{ext gen}} : c = 2, |x| \leq 10$



(d): $u_{4,\text{ext gen}} : c = \frac{1}{2}, |x| \leq 10$

Figure 8: Plots of the additional non-soliton traveling wave solutions to the classical Boussinesq equation via extended generalized tanh method, with $t = 0, 2, 4$. The other solutions are found in Figure 6 and Figure 7.

(a): $u_{7,\text{ext gen}} : c = 2, |x, t| \leq 4$ (b): $u_{8,\text{ext gen}} : c = \frac{1}{2}, |x, t| \leq 4$



(c): $u_{7,\text{ext gen}} : c = 2, |x| \leq 10$ (d): $u_{8,\text{ext gen}} : c = \frac{1}{2}, |x| \leq 10$

Figure 9: Plots of the additional plane periodic solutions to the classical Boussinesq equation via extended generalized tanh method, with $t = 0, 2, 4$. The other solutions are found in Figure 6 and Figure 7.

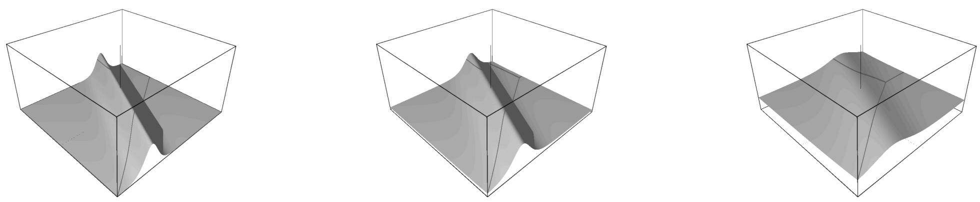
By setting $p = 1$, we validated that the entire set of solutions from the standard and extended standard tanh methods are contained within our generalized framework. This confirms the validity of the ansatz-inspired introductory function Y_p.

4.6. Playing with the paramater p

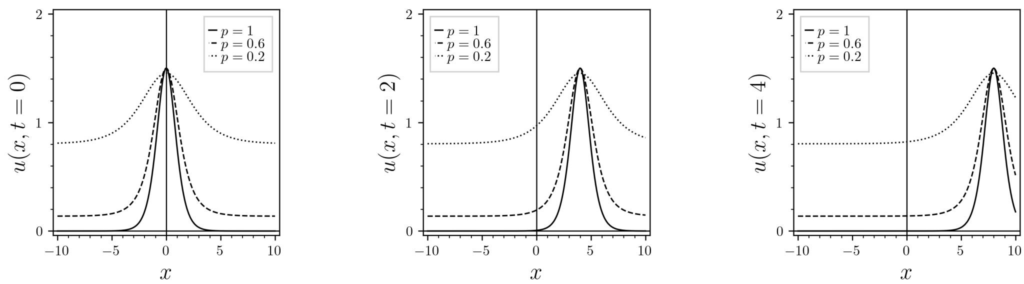
This section explores the specific effects of varying the parameter p on the derived solution families. We use $u_{1,\text{ext gen}}$, $u_{3,\text{ext gen}}$, and $u_{6,\text{ext gen}}$, with $c = 2$, as representative examples of soliton, non-soliton traveling wave, and plane periodic solutions, respectively. This highlights the control we have over the solutions by using p as our tunable parameter.

For $0 \leq p \leq 1$, we observe broadening and amplitude reduction. We saw distinct trends as $p \to 0$ from $p = 1$. The soliton u_1 widens and its amplitude decreases, indicating energy delocalization. The wave flattens suggesting a transition towards a plane-wave-like state or dissipation of its solitary wave characteristics. The effect of forcing seems dominant when $p \to 0$.

The non-soliton traveling wave u_3 also widens, and its depth decreases. The localized feature becomes less pronounced, potentially tending towards a constant solution or losing its distinct waveform. This could imply instability or energy dispersion under conditions represented by smaller p.

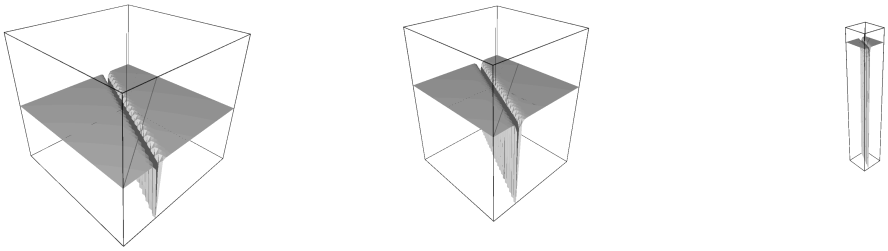


(a): $u_1 : |x,t| \leq 4, p = 1.0$ (b): $u_1 : |x,t| \leq 4, p = 0.6$ (c): $u_1 : |x,t| \leq 4, p = 0.2$

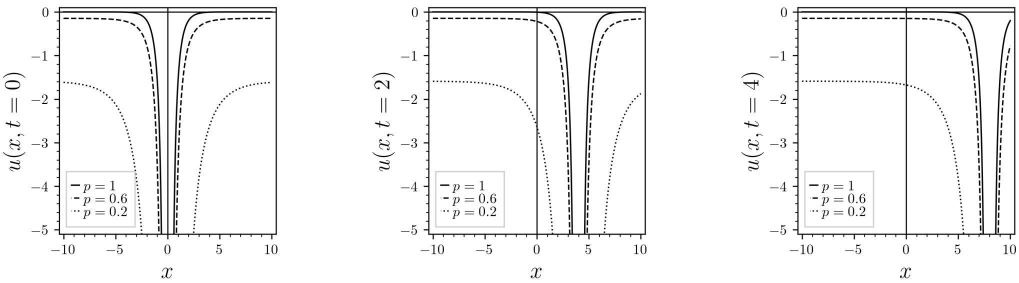


(d): $u_1 : |x| \leq 10, t = 0$ (e): $u_1 : |x| \leq 10, t = 2$ (f): $u_1 : |x| \leq 10, t = 4$

Figure 10: Spacetime evolutions (top) and time evolutions (bottom) of the soliton solution $u_{1,\text{ext gen}}$ for $c = 2$ and $0 \leq p \leq 1$.

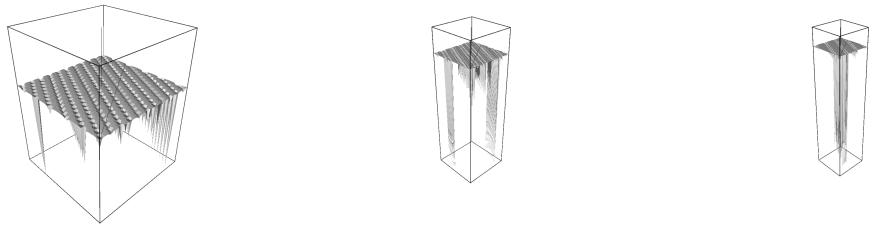


(a): $u_3 : |x,t| \leq 15, p = 1.0$(b): $u_3 : |x,t| \leq 15, p = 0.6$(c): $u_3 : |x,t| \leq 15, p = 0.2$

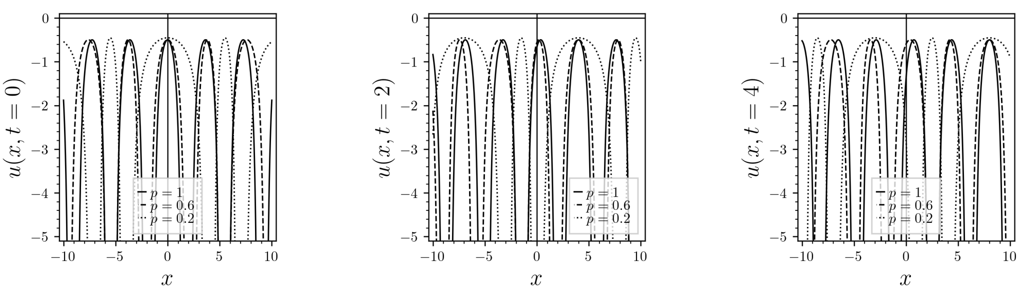


(d): $u_3 : |x| \leq 10, t = 0$ (e): $u_3 : |x| \leq 10, t = 2$ (f): $u_3 : |x| \leq 10, t = 4$

Figure 11: Spacetime evolutions (top) and time evolutions (bottom) of the non-soliton traveling wave solution $u_{3,\text{ext gen}}$ for $c = 2$ and $0 \leq p \leq 1$.



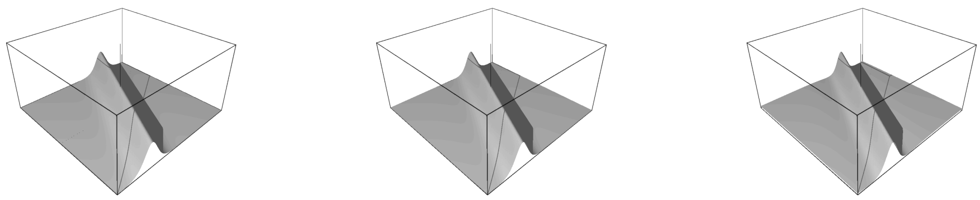
(a): $u_6 : |x,t| \leq 1000, p =$ 1.0 (b): $u_6 : |x,t| \leq 1000, p =$ 0.6 (c): $u_6 : |x,t| \leq 1000, p =$ 0.2



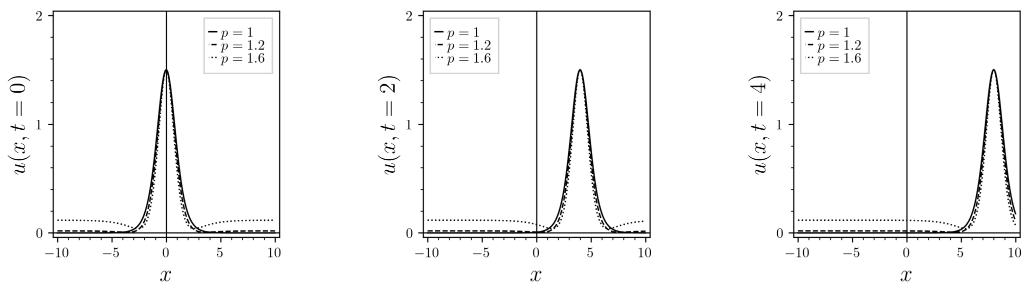
(d): $u_6 : |x| \leq 10, t = 0$ (e): $u_6 : |x| \leq 10, t = 2$ (f): $u_6 : |x| \leq 10, t = 4$

Figure 12: Spacetime evolutions (top) and time evolutions (bottom) of the plane periodic solution $u_{6,\text{ext gen}}$ for $c = 2$ and $0 \leq p \leq 1$.

Whereas, the plane periodic wave u_6 experiences changes in its internal structure within each period. The amplitude and wavelength are modulated, revealing a richer variety of periodic patterns compared to the fixed $p = 1$ case. The relaxation at the origin and compression in surrounding regions illustrate this structural tuning. This dynamic can be likened to a spring mechanism embedded within the solution along the longitudinal axis.



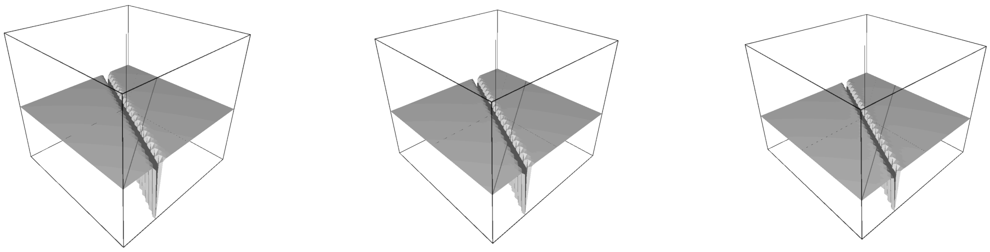
(a): $u_1 : |x, t| \leq 4, p = 1.0$ (b): $u_1 : |x, t| \leq 4, p = 1.2$ (c): $u_1 : |x, t| \leq 4, p = 1.6$



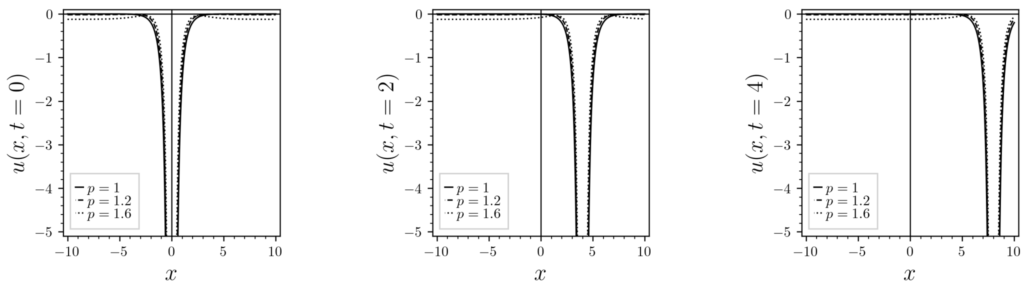
(d): $u_1 : |x| \leq 10, t = 0$ (e): $u_1 : |x| \leq 10, t = 2$ (f): $u_1 : |x| \leq 10, t = 4$

Figure 13: Spacetime evolutions (top) and time evolutions (bottom) of the soliton solution $u_{1,\text{ext gen}}$ for $c = 2$ and $p > 1$.

For $p > 1$, we observe intensification and concentration, an opposite trend. As p increases beyond 1, the soliton u_1 becomes narrower and its amplitude increases, signifying energy concentration and a more sharply peaked wave. The valley in the non-soliton traveling wave u_3 deepens and narrows, making the feature more pronounced and localized. Also, the amplitude of oscillations increases in the plane periodic wave u_6, and the wavelength might be affected, leading to more pronounced periodic variations.

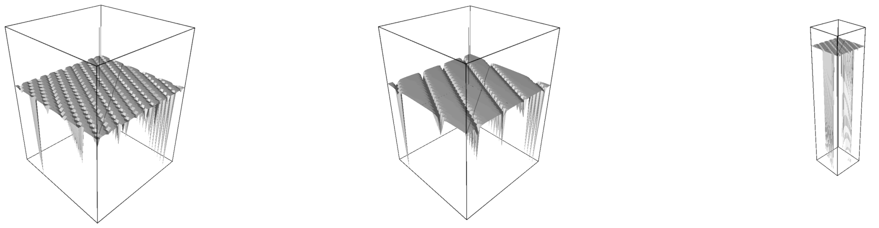


(a): $u_3 : |x,t| \leq 15, p = 1.0$ (b): $u_3 : |x,t| \leq 15, p = 1.2$ (c): $u_3 : |x,t| \leq 15, p = 1.6$

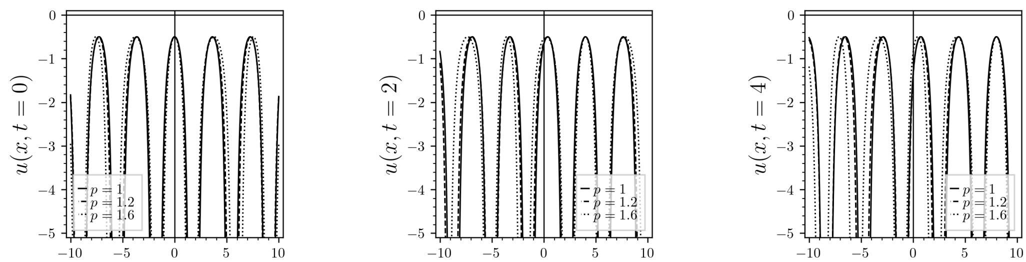


(d): $u_3 : |x| \leq 10, t = 0$ (e): $u_3 : |x| \leq 10, t = 2$ (f): $u_3 : |x| \leq 10, t = 4$

Figure 14: Spacetime evolutions (top) and time evolutions (bottom) of the non-soliton traveling wave solution $u_{3,\text{ext gen}}$ for $c = 2$ and $p > 1$.



(a): $u_6 : |x,t| \leq 1000, p = 1.0$ (b): $u_6 : |x,t| \leq 1000, p = 1.2$ (c): $u_6 : |x,t| \leq 1000, p = 1.6$

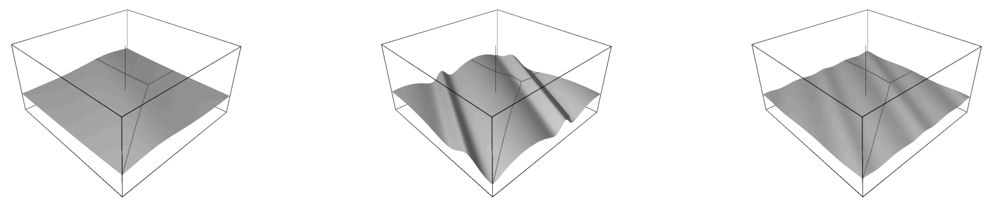


(d): $u_6 : |x| \leq 10, t = 0$ (e): $u_6 : |x| \leq 10, t = 2$ (f): $u_6 : |x| \leq 10, t = 4$

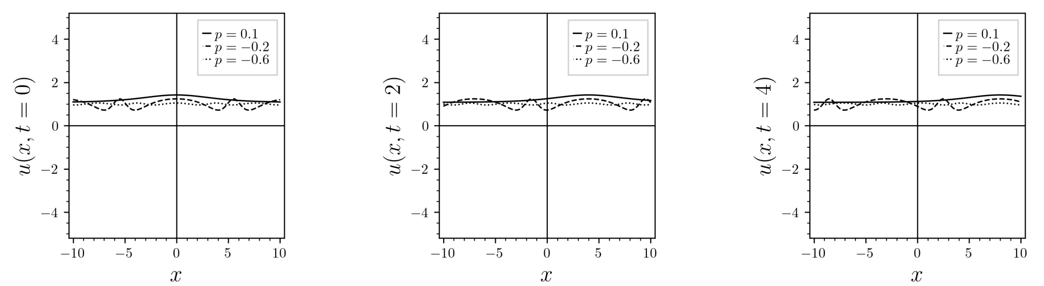
Figure 15: Spacetime evolutions (top) and time evolutions (bottom) of the plane periodic solution $u_{6,\text{ext gen}}$ for $c = 2$ and $p > 1$.

In this regime, localized waves become more intense and compact. This might correspond to scenarios with stronger nonlinearity or different dispersive properties, effectively controlled by p via the forcing term.

For $p < 1$, we observe oscillatory regimes and potential singularities. The regime is fundamentally different because $\mu \propto \sqrt{p}$ becomes imaginary for $c^2 > 1$. This transforms hyperbolic functions in Y_p into trigonometric functions, leading to drastically different solution characteristics. The soliton loses its single-hump shape, becoming oscillatory. It no longer fits the classical soliton definition. The non-soliton traveling wave also transforms into an oscillatory pattern. The Y_p^{-2} term might lead to singularities if $Y_p = 0$, which is possible with trigonometric forms. The plots suggest complex, potentially singular behavior. The plane periodic solution remains periodic but with a significantly altered waveform, often more complex with sharper features or additional oscillations within each period.



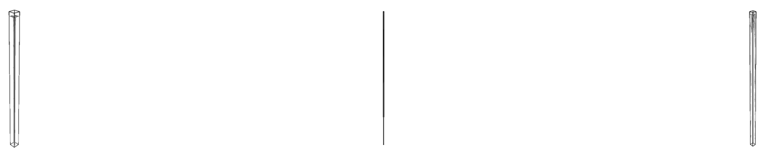
(a): $u_1 : |x, t| \leq 4, p = 0.1$ (b): $u_1 : |x, t| \leq 4, p = -0.2$ (c): $u_1 : |x, t| \leq 4, p = -0.6$



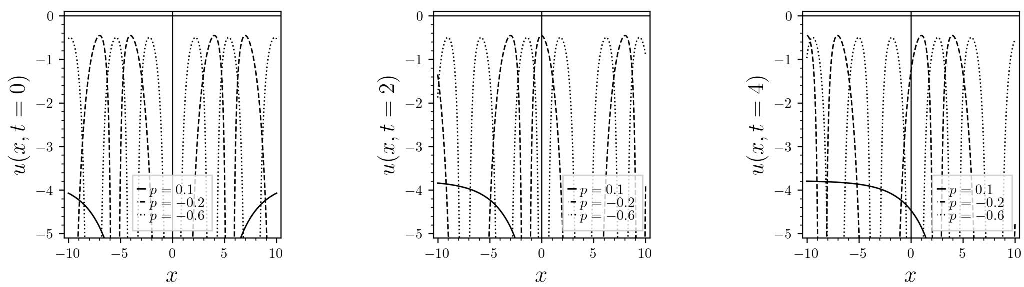
(d): $u_1 : |x| \leq 10, t = 0$ (e): $u_1 : |x| \leq 10, t = 2$ (f): $u_1 : |x| \leq 10, t = 4$

Figure 16: Spacetime evolutions (top) and time evolutions (bottom) of the soliton solution $u_{1,\text{ext gen}}$ for $c = 2$ and $p < 1$.

This regime reveals a rich landscape of oscillatory solutions, as a consequence of the transformation from hyperbolic to trigonometric character (a direct mathematical consequence of p being negative). These solutions are qualitatively different from the $p \geq 0$ cases and might describe entirely different physical phenomena or mathematical structures. The potential for singularities in some of these solutions warrants careful mathematical scrutiny.

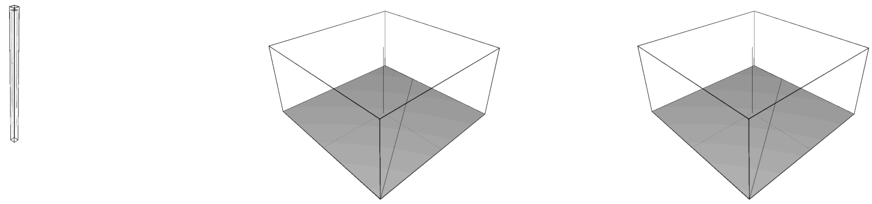


(a): $u_3 : |x, t| \leq 15, p = 0.1$ (b): $u_3 : |x, t| \leq 15, p = -0.2$ (c): $u_3 : |x, t| \leq 15, p = -0.6$

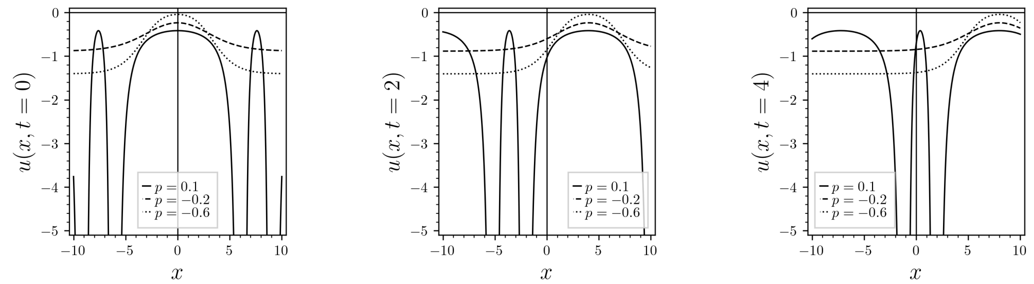


(d): $u_3 : |x| \leq 10, t = 0$ (e): $u_3 : |x| \leq 10, t = 2$ (f): $u_3 : |x| \leq 10, t = 4$

Figure 17: Spacetime evolutions (top) and time evolutions (bottom) of the non-soliton traveling wave solution $u_{3,\text{ext gen}}$ for $c = 2$ and $p < 1$.



(a): $u_6 : |x,t| \leq 1000, p = 0.1$ (b): $u_6 : |x,t| \leq 1000, p = -0.2$ (c): $u_6 : |x,t| \leq 1000, p = -0.6$



(d): $u_6 : |x| \leq 10, t = 0$ (e): $u_6 : |x| \leq 10, t = 2$ (f): $u_6 : |x| \leq 10, t = 4$

Figure 18: Spacetime evolutions (top) and time evolutions (bottom) of the plane periodic solution $u_{6,\text{ext gen}}$ for $c = 2$ and $p < 1$.

5. Conclusions and Recommendations

In this thesis, we have introduced a novel generalization and further extension of the tanh-function method and successfully applied it to the classical Boussinesq equation. This endeavor has yielded a rich spectrum of new, exact, and tunable solutions. The core contributions and findings are

1. Methodological advancement. We developed a generalized tanh-function method, characterized by an ansatz Y_p involving a tunable parameter p. This was further enhanced by an extension incorporating negative powers in the series solution, leading to the extended generalized tanh-function method. The necessary derivative operators for Y_p were rigorously computed forming the mathematical backbone of this generalization.
2. Derivation of new solutions. Application of these methods to the Boussinesq equation resulted in 8 unique families of exact solutions. These encompass tunable solitons, tunable non-soliton traveling waves, and tunable plane periodic solutions. The solutions are tunable in the sense that their qualitative and quantitative features such as amplitude, width, wavelength, and even fundamental form can be continuously varied by adjusting the parameter p.
3. Forcing function and scope. A critical insight is that for $p \neq 1$, the derived solutions satisfy a forced Boussinesq equation, where the forcing term $F(Y_p)$ is explicitly dependent on p and the solution form Y_p. Only when $p = 1$ does this forcing term vanish, and our methods correctly retrieve the known solutions of the standard (unforced) Boussinesq equation obtainable via the standard and extended standard tanh methods. This demonstrates the consistency and broader scope of our generalized approach.
4. Impact of parameter p. The parameter p was shown to be a powerful control mechanism.
   - For $0 \leq p \leq 1$, decreasing p typically leads to wider, flatter localized waves and structurally modulated periodic waves.

- For $p > 1$, localized solutions tend to become narrower and more sharply peaked.
- For $p < 0$, the solutions undergo a fundamental transformation from hyperbolic to trigonometric character due to the wave number μ becoming imaginary. This results in diverse oscillatory patterns, a departure from classical soliton forms, and the potential emergence of singularities.

5. Expansion of solution space. The developed methods significantly expand the known analytical solution space for Boussinesq-type equations. The tunability offered by p provides a framework for generating solution families rather than isolated solutions which can be invaluable for theoretical modeling and understanding the diverse dynamics admitted by such nonlinear systems.

In essence, this work provides a systematic and powerful extension to established analytical techniques for nonlinear partial differential equations offering new avenues for exploring their complex solution landscapes.

The findings of this research open several avenues for future investigation and development.

1. Broader application. Apply the extended generalized tanh-function method to a wider range of nonlinear partial differential equations relevant in physics and engineering, including other types of Boussinesq equations, KdV-type equations, nonlinear Schrödinger equations, and systems in higher dimensions.
2. Exploration of parameter space. Conduct a more exhaustive investigation of the parameter p, including complex values, to explore an even broader class of solutions and their mathematical properties. The physical realizability and stability of solutions for $p < 0$ or complex p warrant detailed study.
3. Analysis of forcing function. Investigate the nature of the forcing function $F\left(Y_p\right)$ that arises when $p \neq 1$. Future work could focus on
   - finding conditions other than $p = 1$ under which $F\left(Y_p\right)$ might vanish, potentially yielding new exact solutions to the unforced equations, and

- exploring physical systems where such forcing terms might naturally arise or can be meaningfully interpreted.

4. Generalization using Riccati equation. Work towards formulating a more encompassing method possibly based on the Riccati equation that could unify various tanh-based methods, including the generalization presented here as special cases. This could lead to a more systematic way of generating diverse classes of exact solutions.
5. Approximate solutions. Explore the potential of this generalized framework for constructing approximate solutions in cases where exact solutions are intractable. The tunable parameter might offer flexibility in optimizing approximations.

Further research in these directions will not only enhance our understanding of nonlinear wave phenomena but also expand the toolkit available for solving and analyzing complex nonlinear systems.

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Appendices

5.1. End-to-end code implementation of the standard tanh method to the classical Boussinesq equation

```
mu, x, c, t, p, Y, Z = s.var("mu x c t p Y Z")
a0, a1, a2 = s.var("a0 a1 a2")


def make_series(m):
    a = [s.var(f"a{i}") for i in range(m + 1)]
    return s.sum(a[i] * Y**i for i in range(m + 1))


def dz(U):
    return mu * (1 - Y**2) * s.diff(U, Y)


def dzz(U):
    return -2 * mu**2 * Y * (1 - Y**2) * s.diff(U, Y) + mu**2 * (
        1 - Y**2
    ) ** 2 * s.diff(s.diff(U, Y), Y)


def flatten(expressions):
    return [expr for expr, _ in expressions]


def prints(*items):
    for item in items:
        print(item)
    print()


U = make_series(2)
ZZ = x - c * t
YY = s.tanh(mu * Z)
UU = U.subs(Y == YY).subs(Z == ZZ)
Beq = (c**2 - 1) * U - 3 * U**2 - dz(dz(U))
seq = Beq.expand().coefficients(Y)
sols = s.solve(flatten(seq), mu, a0, a1, a2)
```

```python
# wazwaz i u1
n = 2
n = 3
prints(
    sols[n], UU.subs(sols[n]),
    UU.subs(sols[n]).full_simplify().trig_reduce(),
)

# wazwaz ii u2
n = 4
n = 5
prints(
    sols[n], UU.subs(sols[n]),
    UU.subs(sols[n]).factor(),
)

# wazwaz i u5
n = 2
n = 3
II = s.sqrt(c**2 - 1) == s.I * s.sqrt(1 - c**2)
prints(
    sols[n], UU.subs(sols[n]).subs(II),
    UU.subs(sols[n]).subs(II).full_simplify().trig_reduce(),
)

# wazwaz ii u8
n = 4
n = 5
II = s.sqrt(1 - c**2) == s.I * s.sqrt(c**2 - 1)
prints(
    sols[n], UU.subs(sols[n]).subs(II),
    UU.subs(sols[n]).subs(II).factor(),
)
```

5.2. Exploratory code implementation of the extended generalized tanh method to the classical Boussinesq equation

```
_u = make_series_ext(2)
_xi = x - c * t
_y = (p + 1) * s.tanh(mu * xi / 2) / (1 + p * s.tanh(mu * xi / 2) ** 2)
u = _u.subs(y == _y).subs(xi == _xi)

Beq = (c**2 - 1) * _u - 3 * _u**2 - dxi_gen(dxi_gen(_u))
seq = Beq.expand().coefficients(y)
sols = s.solve(flatten(seq), mu, a0, a1, a2, b1, b2)

prints(_u, u)
prints(Beq, seq, sols)

seq_qr = seq.copy()
qq = q == (p + 1) / (2 * s.sqrt(p))
rr = r == (p - 1) / (2 * s.sqrt(p))
for i, se in enumerate(seq_qr):
    seq_qr[i][0] = se[0].subs(qq).subs(rr)
sols_qr = s.solve(flatten(seq_qr), mu, a0, a1, a2, b1, b2)

prints(seq_qr, sols_qr)
```

5.3. Exploratory code for investigating parameter p in the solution representatives of the classical Boussinesq equation, derived using the extended generalized tanh method

```
n=7
n=9
xx=4
cc=2

Ug =
UU.subs(sols[n]).subs(c=cc).subs(QQ).subs(RR).simplify().subs(p==1)
P = plot3d(Ug, (-xx,xx), (-xx,xx), color='white', aspect_ratio=1,
plot_points=200)
P = P + axes(xx, color='black')
P.save('../src/fig/B-u1-p=+1.0.png', figsize=[10,10])
P.show()

Ug =
UU.subs(sols[n]).subs(c=cc).subs(QQ).subs(RR).simplify().subs(p==0.6)
P = plot3d(Ug, (-xx,xx), (-xx,xx), color='white', aspect_ratio=1,
plot_points=200)
P = P + axes(xx, color='black')
P.save('../src/fig/B-u1-p=+0.6.png', figsize=[10,10])
P.show()

Ug =
UU.subs(sols[n]).subs(c=cc).subs(QQ).subs(RR).simplify().subs(p==0.2)
P = plot3d(Ug, (-xx,xx), (-xx,xx), color='white', aspect_ratio=1,
    plot_points=200)
P = P + axes(xx, color='black')
P.save('../src/fig/B-u1-p=+0.2.png', figsize=[10,10])
P.show()

xx=10
tt=[1,.6,.2]
ll=['$p=1$', '$p=0.6$', '$p=0.2$']
```

```
yy=2
yy0=0

Ug =
UU.subs(sols[n]).subs(c=cc).subs(QQ).subs(RR).simplify().subs(t==0)
Q = plot([Ug.subs(p=t) for t in tt], (-xx,xx),
    color='black', linestyle=['-','--',':','-.'],
    axes_labels=['$x$','$u(x,t=0)$'], legend_label=ll,
    ticks_integer=True, frame=True,
    typeset='latex', ymin=yy0, ymax=yy)
Q.set_legend_options(shadow=False)
Q.save('../src/fig/B-u1-p0@t=0.png', figsize=[3,3], dpi=300)
Q.show()

Ug =
UU.subs(sols[n]).subs(c=cc).subs(QQ).subs(RR).simplify().subs(t==2)
Q = plot([Ug.subs(p=t) for t in tt], (-xx,xx),
    color='black', linestyle=['-','--',':','-.'],
    axes_labels=['$x$','$u(x,t=2)$'], legend_label=ll,
    ticks_integer=True, frame=True,
    typeset='latex', ymin=yy0, ymax=yy)
Q.set_legend_options(shadow=False)
Q.save('../src/fig/B-u1-p0@t=2.png', figsize=[3,3], dpi=300)
Q.show()

Ug =
UU.subs(sols[n]).subs(c=cc).subs(QQ).subs(RR).simplify().subs(t==4)
Q = plot([Ug.subs(p=t) for t in tt], (-xx,xx),
    color='black', linestyle=['-','--',':','-.'],
    axes_labels=['$x$','$u(x,t=4)$'], legend_label=ll,
    ticks_integer=True, frame=True,
    typeset='latex', ymin=yy0, ymax=yy)
Q.set_legend_options(shadow=False)
Q.save('../src/fig/B-u1-p0@t=4.png', figsize=[3,3], dpi=300)
Q.show()
```

5.4. Computation of derivatives d_ξ, d_ξ^2 by Domingo, Dingel and Bennett (2024)

The first-order differential operator d_ξ is obtained as follows. Let

$$\tanh\frac{\xi}{2} = \frac{1}{\sqrt{p}}\tanh\frac{\Omega}{2} \tag{5.1}$$

such that the g-HATH ansatz becomes

$$\begin{aligned}
Y_p(\xi) &= (1+p)\frac{\tanh\frac{\xi}{2}}{1+p\tanh^2\frac{\xi}{2}} \\
&= \frac{1+p}{\sqrt{p}}\frac{\tanh\frac{\Omega}{2}}{1+\tanh^2\frac{\Omega}{2}} \\
&= \frac{1+p}{2\sqrt{p}}\frac{2\tanh\frac{\Omega}{2}}{1+\tanh^2\frac{\Omega}{2}} \\
&= \frac{1+p}{2\sqrt{p}}\tanh\Omega \\
&= Y_p(\Omega).
\end{aligned} \tag{5.2}$$

Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi} &= \frac{\mathrm{d}Y_p}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}Y_p} = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\xi}\frac{\mathrm{d}Y_p}{\mathrm{d}\Omega}\right)\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(2\tanh^{-1}\left(\sqrt{p}\tanh\frac{\xi}{2}\right)\right)\frac{\mathrm{d}}{\mathrm{d}\Omega}\left(\frac{1+p}{2\sqrt{p}}\tanh\Omega\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(\frac{2\sqrt{p}}{1-\left(\sqrt{p}\tanh\frac{\xi}{2}\right)^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\tanh\frac{\xi}{2}\right)\right)\left(\frac{1+p}{2\sqrt{p}}(1-\tanh^2\Omega)\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{1+p}{2}\frac{(1-\tanh^2\Omega)\left(1-\tanh^2\frac{\xi}{2}\right)}{1-p\tanh^2\frac{\xi}{2}}\right]\frac{\mathrm{d}}{\mathrm{d}Y_p}.
\end{aligned} \tag{5.3}$$

Recall the identities

$$\begin{aligned}
&\tanh\Omega = \frac{2\tanh\frac{\Omega}{2}}{1+\tanh^2\frac{\Omega}{2}} \Longrightarrow \tanh^2\Omega = \frac{4\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}, \\
&\tanh\frac{\xi}{2} = \frac{1}{\sqrt{p}}\tanh\frac{\Omega}{2}.
\end{aligned} \tag{5.4}$$

Our expression can then be expressed entirely in terms of $\frac{\Omega}{2}$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi} &= \left[\frac{1+p}{2}\frac{\left(1-\frac{4\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}\right)\left(1-\frac{1}{p}\tanh^2\frac{\Omega}{2}\right)}{1-\tanh^2\frac{\Omega}{2}}\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{1+p}{2}-\left(\frac{1+p}{p}\frac{2\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}\frac{p-\tanh^2\frac{\Omega}{2}}{1-\tanh^2\frac{\Omega}{2}}\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{2}{1+p}\left(\frac{(1+p)^2}{4p}-\frac{(1+p)^2}{p}\frac{\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}\right)\left(\frac{p-\tanh^2\frac{\Omega}{2}}{1-\tanh^2\frac{\Omega}{2}}\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{1+p}{2}\frac{\left(1-\frac{4\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}\right)\left(1-\frac{1}{p}\tanh^2\frac{\Omega}{2}\right)}{1-\tanh^2\frac{\Omega}{2}}\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{1+p}{2p}-\frac{1+p}{p}\frac{2\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}\left(\frac{p-\tanh^2\frac{\Omega}{2}}{1-\tanh^2\frac{\Omega}{2}}\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{2}{1+p}\left(\frac{(1+p)^2}{4p}-\frac{(1+p)^2}{p}\frac{\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}\right)\left(\frac{p-\tanh^2\frac{\Omega}{2}}{1-\tanh^2\frac{\Omega}{2}}\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\frac{2}{1+p}\left(\frac{(1+p)^2}{4p}-\frac{(1+p)^2}{4p}\tanh^2\Omega\right)\left(\frac{p-\tanh^2\frac{\Omega}{2}}{1-\tanh^2\frac{\Omega}{2}}\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p}. \qquad (5.5)
\end{aligned}
$$

Recall that

$$
\begin{aligned}
q_p &= \frac{1+p}{2\sqrt{p}} \Longrightarrow q_p^2 = \frac{(1+p)^2}{4p} \\
Y_{p(\Omega)} &= \frac{1+p}{2\sqrt{p}}\tanh\Omega \Longrightarrow Y_p^2 = \frac{(1+p)^2}{4p}\tanh^2\Omega. \qquad (5.6)
\end{aligned}
$$

Substituting, our expression becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi} &= \left[\frac{2}{1+p}\left(q_p^2-Y_p^2\right)\frac{p-\tanh^2\frac{\Omega}{2}}{1-\tanh^2\frac{\Omega}{2}}\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2-Y_p^2\right)\left(1-1+\frac{2p-2\tanh^2\frac{\Omega}{2}}{(1+p)\left(1-\tanh^2\frac{\Omega}{2}\right)}\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2-Y_p^2\right)\left(1+\frac{-(1+p)\left(1-\tanh^2\frac{\Omega}{2}\right)+2p-2\tanh^2\frac{\Omega}{2}}{(1+p)\left(1-\tanh^2\frac{\Omega}{2}\right)}\right)\right]\frac{\mathrm{d}}{\mathrm{d}Y_p} \qquad (5.7)
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\left(q_p^2 - Y_p^2\right) \left(1 + \frac{(p-1)\left(1+\tanh^2\frac{\Omega}{2}\right)}{(1+p)\left(1-\tanh^2\frac{\Omega}{2}\right)} \right) \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2 - Y_p^2\right) \left(1 + \frac{p-1}{1+p} \sqrt{\frac{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}{\left(1-\tanh^2\frac{\Omega}{2}\right)^2}} \right) \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2 - Y_p^2\right) \left(1 + \frac{p-1}{1+p} \sqrt{\frac{1}{\frac{\left(1-\tanh^2\frac{\Omega}{2}\right)^2}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}}} \right) \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2 - Y_p^2\right) \left(1 + \frac{p-1}{1+p} \sqrt{\frac{1}{1 - \frac{4\tanh^2\frac{\Omega}{2}}{\left(1+\tanh^2\frac{\Omega}{2}\right)^2}}} \right) \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2 - Y_p^2\right) \left(1 + \frac{p-1}{1+p} \frac{1}{\sqrt{\frac{4p}{(p+1)^2}\left(\frac{(p+1)^2}{4p} - \frac{(p+1)^2}{4p}\tanh^2\Omega\right)}} \right) \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2 - Y_p^2\right) \left(1 + \frac{p-1}{1+p} \frac{p+1}{2\sqrt{p}} \frac{1}{\sqrt{q_p^2 - Y_p^2}} \right) \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2 - Y_p^2\right) \left(1 + \frac{r_p}{\sqrt{q_p^2 - Y_p^2}} \right) \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \\
&= \left[\left(q_p^2 - Y_p^2\right) + r_p\sqrt{q_p^2 - Y_p^2} \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \qquad (5.7)
\end{aligned}
$$

where $r_p = \frac{p-1}{2\sqrt{p}}$.

The second-order differential operator d_ξ^2 is obtained as follows

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} &= \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right) \\
&= \left[\left(q_p^2 - Y_p^2\right) + r_p\sqrt{q_p^2 - Y_p^2} \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \left[\left[\left(q_p^2 - Y_p^2\right) + r_p\sqrt{q_p^2 - Y_p^2} \right] \frac{\mathrm{d}}{\mathrm{d}Y_p} \right] \\
&= \left[\left(q_p^2 - Y_p^2\right) + r_p\sqrt{q_p^2 - Y_p^2} \right] \left(-2Y_p - \frac{r_p Y_p}{\sqrt{q_p^2 - Y_p^2}} \right) \\
&\quad + \left[\left(q_p^2 - Y_p^2\right) + r_p\sqrt{q_p^2 - Y_p^2} \right]^2 \frac{\mathrm{d}^2}{\mathrm{d}Y_p^2}. \qquad (5.8)
\end{aligned}
$$