Interference of two circularly polarized far-field dipole radiation: energy and momentum density computations via geometric algebra

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Introduction

The phenomenon of interference in two-source systems is a very old problem. One of its groundbreaking iterations, the well-known two-slit setup by Young, is now two centuries old [1]. The two-slit interference not only was able to lead us to our current understanding of the nature of light [2], it is also serving as an important conceptual tool nowadays in explorations done in both the classical [3] and quantum [4] domains. Since its success, many iterations have been crafted and have been actively studied such as two-pinhole systems, and two-slit systems with electrons [5], photons [6] and even atoms [7]. Though to wit, many-source interference (or more appropriately, diffraction) systems are also an active field and are as equally as insight-rich as the original two-source system which served as its inspiration.

In two-source interference papers employing electromagnetic waves as sources, one usually overlooks the polarimetric properties of the waves made to interfere. The results they communicate are then responsive only to the case when the waves are assumed to be unpolarized. They do not always have exact correspondence with the case of otherwise polarized waves. There are works such as [8], [9] and [10] which deal with circularly polarized sources, or even elliptical ones, and whose results we will consider in the later sections. There are also application-focused works in polarization optics which are usually based on the use of circularly polarized light. Such works include microparticle rotations [11], optical fibers [12], and turbulent medium imaging [13]. However, most of them considered polarized but planar waves, or more precisely beam-like or paraxial, which can qualify as a special case to spherical waves.

In this paper, we revisit the two-source interference problem by employing two circularly polarized electric dipoles as sources. We formulate the electromagnetic interference by establishing an expression for the dipole in the far-field radiation zone via geometric algebra. This expression describes the electromagnetic field as a unified mathematical object in the form of the sum of electric field and imaginary magnetic field in spherical coordinates. We then calculate the energy and momentum densities resulting from the interference. The computation allows us to construct the intensity profile and interference pattern of the two-dipole system which are then to be compared to the findings in the interference literature.

This paper is essentially an extension of the geometric algebra formulation by [10]. The electromagnetic wave in that formulation is propagating through planar wave fronts, whereas in this paper, it propagates through spherical wave fronts. We may verify the results of [10] by approximating the spherical wave fronts as planar since the former is locally flat geometrically speaking. The scope of this paper restricts us to the formulation of the electromagnetic field expression only for the circularly polarized case. However, a possible extension to the more general elliptical case is being considered. We also restrict to the electromagnetic waves propagating through the far-field region of the dipole as the near-field zone contains non-radiative terms in the electromagnetic field expression.

Methodology

In this section, we show how the interfering electromagnetic fields in a two-dipole system of circular polarization is formulated in terms of geometric algebra. We demonstrate how the energy and momentum densities resulting from the interference are parametrized and computed. We also describe how the intensity profiles and interference patterns with varied parameters are constructed from the electromagnetic field formulation which are to be compared to the findings in general literature.

We start with the expressions for electric field intensity $\mathbf{E}$ and magnetic flux density $\mathbf{B}$ of an electric dipole radiation in the far-field zone given by Griffiths in [14] which are

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} = E_0 e^{i\omega(r/c-t)} \mathbf{e}_\theta \quad \text{and} \tag{1}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{c} E_0 e^{i\omega(r/c-t)} \mathbf{e}_\phi, \tag{2}$$

where

$$E_0 \equiv \frac{-\mu_0 p_0 \omega^2}{4\pi}. \tag{3}$$

Since $\mathbf{B} = \mu_0 \mathbf{H}$ for the magnetic field intensity $\mathbf{H}$ in free space, then we can express (2) as

$$\mathbf{H} = \boldsymbol{\nabla} \times \frac{\mathbf{A}}{\mu_0} = \frac{1}{\mu_0 c} E_0 e^{i\omega(r/c-t)} \mathbf{e}_\phi. \tag{4}$$

Using (1) and (4), we now construct the electromagnetic field expression $\hat{\mathbf{E}}$ as the sum of $\mathbf{E}$ and $\mathbf{H}$ as in

$$\hat{\mathbf{E}} = \mathbf{E} + i\zeta \mathbf{H}, \tag{5}$$

or more explicitly,

$$\hat{\mathbf{E}} = (\mathbf{e}_\theta + i\mathbf{e}_\phi) E_0 e^{i\omega(r/c-t)}, \tag{6}$$

where $\zeta = \sqrt{\mu_0/\epsilon_0}$ is the radiation resistance, and $i = \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\phi$ is the orthonormal trivector in Clifford Algebra. This expression allows us to describe a circularly polarized electromagnetic field using the exponential function $e^{i\omega(r/c-t)}$. This is in comparison to the electric and magnetic field expressions in [14] and other vector calculus-based books wherein only the real parts of $\mathbf{E}$ and $\mathbf{H}$ are set to have a physical correspondence in the radiative phenomenon thus yielding a linearly polarized field as a result.

Then, we show that $\hat{\mathbf{E}}$ in (6) corresponds to Maxwell's equations in free space by letting the spacetime derivative of $\hat{\mathbf{E}}$ to vanish, that is

$$\frac{\partial \hat{\mathbf{E}}}{\partial \hat{r}} = 0, \tag{7}$$

where

$$\frac{\partial}{\partial \hat{r}} = \boldsymbol{\nabla} + \frac{1}{c} \frac{\partial}{\partial t} \tag{8}$$

is the spacetime derivative operator. Expanding (7) via (8) and using the geometric definition of vectorial product given by

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}) \tag{9}$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$, then we expect to get

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \tag{10}$$

$$\boldsymbol{\nabla} \times \zeta \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \tag{11}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c} \frac{\partial \zeta \mathbf{H}}{\partial t}, \tag{12}$$

$$\text{and} \quad \boldsymbol{\nabla} \cdot \zeta \mathbf{H} = 0 \tag{13}$$

after grouping the terms according to their vectorial grades. These equations are Gauss's law, Ampere's law, Faraday's law, and the magnetic flux continuity law, respectively.

We remark that [15] showed that one of the solutions to this one-equation form of Maxwell's equation in (7) is a circularly polarized electromagnetic wave, which is the condition exemplified by $\hat{\mathbf{E}}$ in (6).

Once evaluated and confirmed that (7) holds for $\hat{\mathbf{E}}$ in (6), we then compute the energy density U and momentum density $\mathbf{S}/c$ of the electric dipole through

$$\frac{\hat{\mathbf{S}}}{c} \equiv U + \frac{\mathbf{S}}{c} = -\frac{1}{2}\epsilon_0 \hat{\mathbf{E}}\hat{\mathbf{E}}^\dagger, \tag{14}$$

where $\hat{\mathbf{E}}^\dagger$ is the spatial inverse of $\hat{\mathbf{E}}$ and explicitly is

$$\hat{\mathbf{E}}^\dagger = (\mathbf{E} + i\zeta\mathbf{H})^\dagger = -\mathbf{E} + i\zeta\mathbf{H}. \tag{15}$$

Meanwhile, we consider two electric dipoles which are pointing through the positive z-axis and are separated by horizontal distance d. We reconcile the resulting non-connected coordinate systems carried by each of the said dipoles by making $d/2$ the origin of a new primed spherical coordinate system. As a result, we can then make the fields of each of the dipole interfere with themselves.

Now that the necessary expressions $\hat{\mathbf{E}}$ and $\hat{\mathbf{S}}/c$ describing the radiative phenomenon of an electric dipole were derived in (6) and (14), respectively, we compute the time-average of the energy-momentum density through

$$\frac{\langle \hat{\mathbf{S}} \rangle}{c} = \frac{1}{\tau}\int_0^\tau \frac{\hat{\mathbf{S}}}{c} dt, \tag{16}$$

where one period of oscillation is $\tau = 2\pi/\omega$. We then separate the resulting expression by vectorial grades to get $\langle U \rangle$ and $\langle \mathbf{S} \rangle /c$. We note that the resulting analog for (16) for this interfering two-dipole system will be

$$\frac{\langle \hat{\mathbf{S}} \rangle}{c} = \frac{\langle \hat{\mathbf{S}}_1 \rangle}{c} + \frac{\langle \hat{\mathbf{S}}_2 \rangle}{c} \tag{17}$$

We plot these time-averaged energy $\langle U \rangle$ and momentum $\langle \mathbf{S} \rangle /c$ densities as functions of r and θ for $\phi = \pi/2$. The energy density plot as a system will be a contour plot, whereas the momentum density plot will be a vector field plot.

We will also plot these as two-slit interference analog wherein $\langle U \rangle$ and $\langle \mathbf{S} \rangle /c$ are plotted against a screen oriented along the positive z-axis which is the same orientation as that of the two-dipole system. The screen will be positioned perpendicular to the d-long horizontal we previously defined and within the far-field zone of an electric dipole. We then set $\theta = \pi/2$ such that the position of an arbitrary point on the screen will be described as a function of x.

Finally, we will compare the intensity profiles on the screen with that of in the literature. Our results will be more general than that of the literature since they make use of linearly polarized sources rather than circular.

Expected Results

In this paper, we shall describe the electromagnetic interference of two circularly polarized electric dipoles situated in the far-field via energy and momentum density computations. For that, we shall establish an expression for the electromagnetic field $\hat{\mathbf{E}}$ of a circularly polarized electric dipole in terms of geometric algebra that is analogous to the formulation of [15] given by

$$\hat{\mathbf{E}}_\pm = (\mathbf{e}_1 + i\mathbf{e}_2)a_\pm e^{\pm i\mathbf{e}_3(\omega t - kz \pm \delta_\pm)} \tag{18}$$

or to that of [16] in

$$\mathbf{F}(s) = ikc\hat{\mathbf{a}}(a_+ e^{-is} + a_- e^{is}), \tag{19}$$

where either a_+ or a_- should vanish for the wave to be circularly polarized, otherwise it is plane polarized if $a_+ = a_-$ or elliptically polarized if not equal and not vanishing. We remark that these formulations assume that the electromagnetic wave described are planar rather than spherical.

We shall also compute for the energy and momentum densities and plot the corresponding intensity profile of the two-dipole system. The plots are expected to be analogous to that in the interference literature.

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