New exact solution families for forced Boussinesq equation via an extension of generalized tanh-function method

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Boussinesq equation

$$\partial_t^2 u - c^2\partial_x^2 u - \alpha\partial_x^2 u^2 - \beta\partial_x^4 u = 0$$

- Is a physically and mathematically interesting nonlinear partial differential equation (pde)
- Models diverse wave phenomena across fluids, plasmas, and materials
- Shows how nonlinearity and dispersion balance to form stable, particle-like waves
- Reveals potential for singularity formation and decay, pushing the boundaries of classical soliton understanding [1]

The idea behind our method

- We exploit $Y(\xi) = \tanh\xi$ and its self-similarity on differentiation

$$\mathrm{d}_\xi Y = \mathrm{sech}^2\,\xi = 1 - Y^2, \quad \mathrm{d}_\xi^2 Y = -2Y + 2Y^3, \quad \mathrm{d}_\xi^3 Y = -2 + 8Y^2 - 6Y^4, \quad \ldots$$

- Then we replace $Y(\xi)$ with this ansatz first presented in [2] and inspired by half-angle identity, where $0 \le p \le 1$, $p \in \mathbb{R}$

$$Y_{p,\xi} = (1+p)\frac{\tanh\frac{\mu\xi}{2}}{1 + p\tanh^2\frac{\mu\xi}{2}}$$

- Finally we extend the solution set to include those based on coth, csch by extending the series [3] to

$$S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}$$

Generalized tanh method and its extension

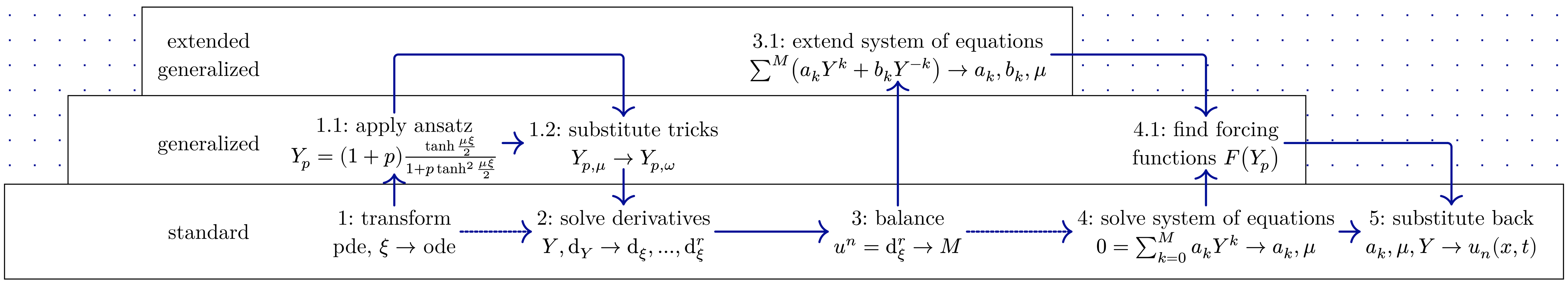


Figure 1: Procedures of the standard (std) tanh method [4], along with our proposed generalization (gen) and subsequent extension (ext gen) of the method.

Results

For the classical Boussinesq equation with $c = 1$, $\alpha = 3$, and $\beta = 1$, we do

1: transform pde + 1.1: apply ansatz

- Using the transformation $\xi = \mu(x - ct)$ and integrating twice, we get

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2(3u^2) - \partial_x^4 u = 0 \implies (c^2-1)u - 3u^2 - \mathrm{d}_\xi^2 u = 0.$$

1.2: substitute tricks + 2: solve derivatives

- Tackling via $\mathrm{d}_\xi = \mathrm{d}_\xi\omega \cdot \mathrm{d}_\omega Y_p \cdot \mathrm{d}_{Y_p}$ and $\mathrm{d}_\xi^2 = \mathrm{d}_\xi \cdot (\mathrm{d}_\xi Y_p \cdot \mathrm{d}_{Y_p})$, we compute

$$\mathrm{d}_\xi = \mu\Big[\big(q_p^2 - Y_{p,\xi}^2\big) + r_p\big(q_p^2 - Y_{p,\xi}^2\big)^{\frac{1}{2}}\Big]\mathrm{d}_{Y_p},$$

$$\mathrm{d}_\xi^2 = \mu^2\Big[\big(q_p^2 - Y_{p,\xi}^2\big) + r_p\big(q_p^2 - Y_{p,\xi}^2\big)^{\frac{1}{2}}\Big]^2\mathrm{d}_{Y_p}^2$$
$$+\mu^2\Big[\big(q_p^2 - Y_{p,\xi}^2\big) + r_p\big(q_p^2 - Y_{p,\xi}^2\big)^{\frac{1}{2}}\Big]\Big[-2Y_{p,\xi} - r_p Y_{p,\xi}\big(q_p^2 - Y_{p,\xi}^2\big)^{-\frac{1}{2}}\Big]\mathrm{d}_{Y_p},$$

where $q_p \equiv (p+1)/(2\sqrt{p})$ and $r_p \equiv (p-1)/(2\sqrt{p}) = (p-1)/(p+1)q_p$.

3: balance + 3.1: extend system of equations

- Balancing the highest-order nonlinear term with highest-order derivative

$$u^2 = \mathrm{d}_\xi^2 u \implies u(Y_{p,\xi}) = a_0 + a_1 Y_{p,\xi} + a_2 Y_{p,\xi}^2 + b_1 Y_{p,\xi}^{-1} + b_2 Y_{p,\xi}^{-2}.$$

4.1: find forcing functions

- This leads to a more extensive system of equations with the terms involving non-integer powers of Y_p, which we already isolate into the forcing function

$$F(Y_p) = \mu^2 r\big[-2b_2q^2Y_p^{-2} - b_1q^2Y_p^{-1} + 2b_2 + (a_1q^2 + b_1)Y_p + 2a_2q^2Y_p^2 - a_1Y_p^3 - 2a_2Y_p^4\big]\big(q_p^2 - Y_p^2\big)^{-\frac{1}{2}}$$
$$+\mu^2 r\big[-12b_2q^2Y_p^{-4} - 4b_1q^2Y_p^{-3} + 8b_2Y_p^{-2} + 2b_1Y_p^{-1} + (-4a_2q^2 - b_1q^2) + 2a_1Y_p + 8a_2Y_p^2\big]\big(q_p^2 - Y_p^2\big)^{\frac{1}{2}}.$$

- We obtain a forced version of the Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u - 3\partial_x^2 u^2 - \partial_x^4 u = F(Y_p).$$

- This modification allowed us to eliminate terms with non-integral powers of Y. Importantly, the original, unforced Boussinesq equation is recovered by setting $p = 1$, which makes $r_p = 0$ and therefore $F(Y_p) = 0$.

5: substitute back

- From an initial set of 14 solutions, we identified 8 unique families: 2 are solitons, 2 are non-soliton traveling waves, and the remaining solutions are plane periodic solutions. u_1, u_3, u_6 are representative solutions per type

$$u_1(x,t,p) = \frac{c^2-1}{6}(1 + P_1P_0^2) + 2(1-c^2)(p+1)^2P_0^2\left[\frac{\tanh\left(P_0\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1 + p\tanh^2\left(P_0\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

$$u_3(x,t,p) = \frac{c^2-1}{6}(1 + P_1P_0^2) + (1-c^2)\frac{p^2+1}{4p^2}P_0^2\left[\frac{1 + p\tanh^2\left(P_0\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{\tanh\left(P_0\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

$$u_6(x,t,p) = \frac{c^2-1}{6}(1 - P_1P_0^2) + 2(1-c^2)(p+1)^2P_0^2\left[\frac{\tan\left(P_0\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1 - p\tan^2\left(P_0\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

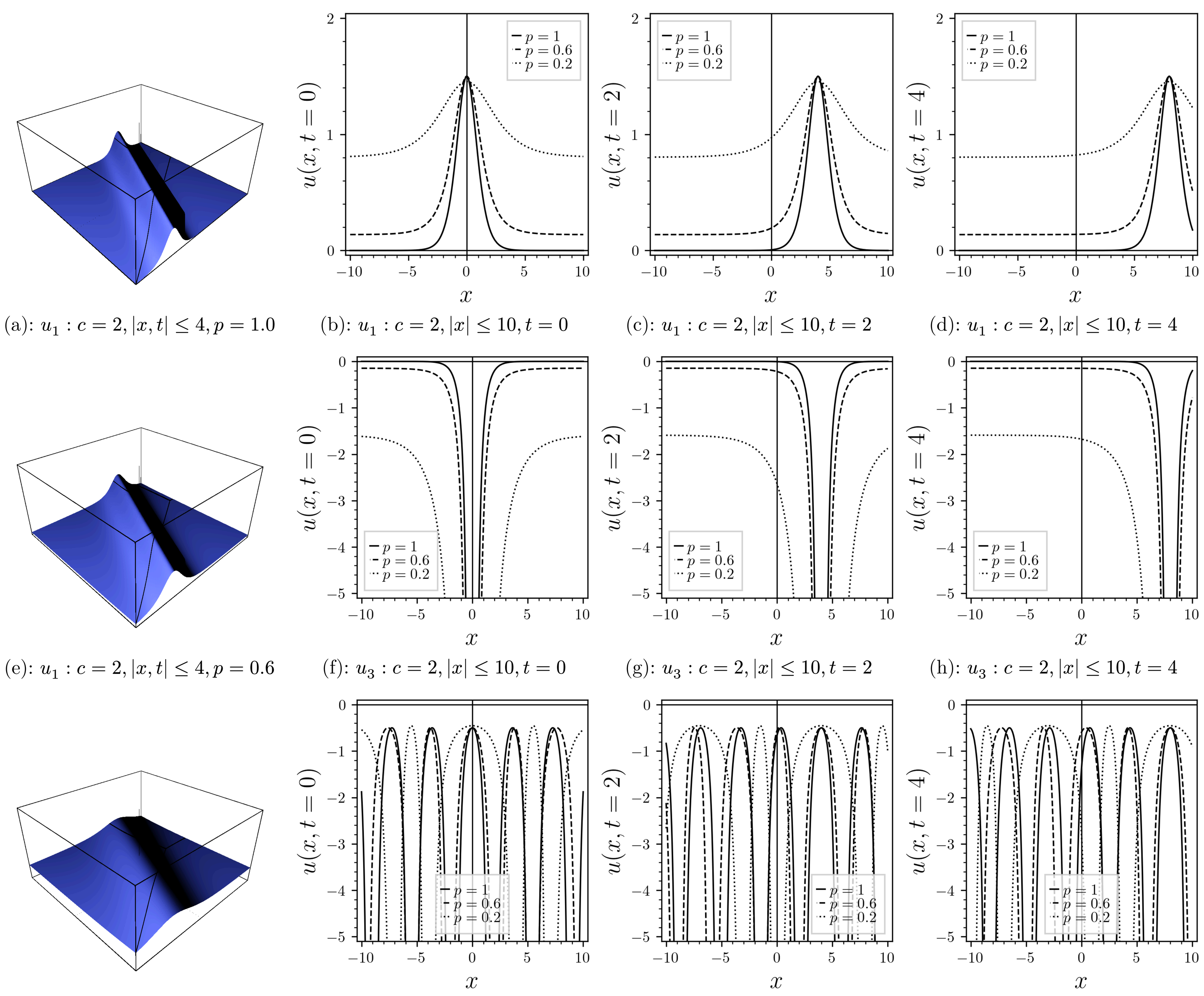
where $P_0 \equiv \sqrt{p}/\sqrt[4]{(3p^2+1)(p^2+3)}$ and $P_1 \equiv (3p^2 + 2p + 3)/p$.

- Quick sanity check: we set the parameter $p = 1$ and get

$$u_1(x,t,p=1) = \frac{c^2-1}{2} + 2(1-c^2)\left[\frac{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1 + \tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2 = \frac{c^2-1}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) = u_1(x,t)_{\mathrm{std}}$$

$$u_3(x,t,p=1) = \frac{c^2-1}{2} + \frac{1-c^2}{8}\left[\frac{1 + \tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2 = -\frac{c^2-1}{2}\mathrm{csch}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) = u_3(x,t)_{\mathrm{ext\ std}}$$

$$u_6(x,t,p=1) = -\frac{c^2-1}{6} + 2(1-c^2)\left[\frac{\tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1 - \tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2 = -\frac{c^2-1}{6}\left[1 + 3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right] = u_4(x,t)_{\mathrm{std}}$$



(a): $u_1 : c = 2, |x,t| \le 4, p = 1.0$ (b): $u_1 : c = 2, |x| \le 10, t = 0$ (c): $u_1 : c = 2, |x| \le 10, t = 2$ (d): $u_1 : c = 2, |x| \le 10, t = 4$

(e): $u_1 : c = 2, |x,t| \le 4, p = 0.6$ (f): $u_3 : c = 2, |x| \le 10, t = 0$ (g): $u_3 : c = 2, |x| \le 10, t = 2$ (h): $u_3 : c = 2, |x| \le 10, t = 4$

(i): $u_1 : c = 2, |x,t| \le 4, p = 0.2$ (j): $u_6 : c = 2, |x| \le 10, t = 0$ (k): $u_6 : c = 2, |x| \le 10, t = 2$ (l): $u_6 : c = 2, |x| \le 10, t = 4$

Figure 2: Spacetime evolutions (a, e, i) of a soliton solution u_1 for different p values. Time evolutions (b, c, d) of a soliton solution u_1, (f, g, h) of a non-soliton traveling wave solution u_3, (j, k, l) of a plane periodic solution u_6, for parameter $p = 0.2, 0.6, 1.0$.

Conclusions

We introduced a novel extension following a generalization of tanh method for nonlinear pdes, featuring a parameter p for tunability. Applied to Boussinesq equation, we identified 8 solution families including solitons, non-soliton traveling waves, and plane periodic wave solutions. We found that our method encompasses standard tanh solutions when $p = 1$.

References

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