Novel exact solutions for forced Boussinesq equation via extended generalized tanh-function method

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1. Introduction

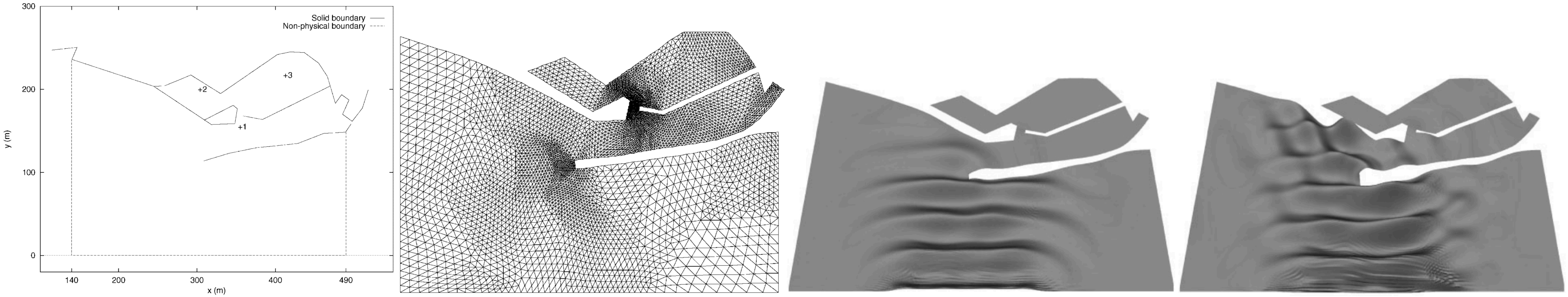
1.1. Boussinesq equation

- Represents one of the most challenging areas in mathematical physics for studying nonlinear wave phenomena
- Originally developed for long waves in shallow water (1870s), now appears in diverse systems including plasmas [1,2], the atmosphere [3,4], acoustic-like regimes [5], dielectrics [6], antiferromagnets [7], and nonlinear strings [8]
- With bidirectional wave propagation, unlike KdV say, has the form

$$\partial_t^2 u - c^2 \partial_x^2 - \alpha \partial_x^2 u^2 - \beta \partial_x^4 u = 0$$

- Closely connected to KdV, Kadomtsev-Petviashvili, and nonlinear Schrödinger equations under various conditions

1.1. Boussinesq equation



(a): Spatial domain (b): Intermediate mesh (c): Model at $t = 30$ (d): Model at $t = 30$

Figure 1: Unstructured triangular meshes of the harbor geometry (a, b) and simulated model of the the waves on the free surface at time t (c, d) [9].

1.1. Boussinesq equation

- Has the following wave physics and characteristics
  - Competing effects. Balance between nonlinearity (wave steepening) and linear dispersion (wave spreading) enables soliton solutions
  - Soliton properties. Particle-like waves with stable profile, constant shape and speed, but can exhibit complex behaviors like singularity formation
  - Frequency dispersion. Accounts for broader range of wave phenomena than classical shallow-water equations
  - Mathematical variants. Different forms exist (well-posed vs ill-posed) depending on parameter β sign, with both classical forms being completely integrable

1.2. Our research

The problem and motivation

- Traditional approaches like standard tanh-function method produce solutions with fixed characteristics, limiting modeling flexibility
- We address this limitation through extended generalized tanh-function method producing tunable solution families through the incorporation of an ansatz with tunable parameter p

1.2. Our research

Research objectives and scope

- Primary goal. Derive new tunable soliton, periodic, and traveling wave solutions for classical Boussinesq equation with $\alpha = 3, \beta = 1$
- Forced equation insight. Demonstrate that solutions for $p \neq 1$ pertain to forced Boussinesq equation with forcing term dependent on parameter p
- Method validation. Show that standard tanh method solutions are recovered as special cases, confirming method consistency
- Parameter influence. Systematically analyze how tunable parameter p affects solution characteristics and physical interpretation

2. Methodology

2.1. Generalization of the tanh method

Building upon the standard tanh-function approach, we replace the traditional introductory function Y with a novel ansatz first presented by Buenaventura, Dingel and Calgo in [10], inspired by the half-angle identity in tanh-function and parametrized by a tunable parameter p

$$Y_{p,\xi} = Y_p(\mu\xi) = (1+p)\frac{\tanh\frac{\mu\xi}{2}}{1+p\tanh^2\frac{\mu\xi}{2}}, \qquad 0 \le p \le 1, \quad p \in \mathbb{R}.$$

The key feature of this ansatz is the tunable parameter p. It could allow for solutions to be either adaptively tailored to the specific problem at

2.1. Generalization of the tanh method

hand or precisely fine-tuned to meet specific conditions. Following Malfliet's approach outlined in the figure above, we transform the pde

$$p(u, \partial_t u, \partial_x u, \partial_t^2 u, \partial_x^2 u, \partial_x \partial_t u, \ldots) = 0$$

into a nonlinear ordinary differential equation

$$P\left(U, \mathrm{d}_\xi U, \mathrm{d}_\xi^2 U, \ldots\right) = 0$$

together with their respective solutions $u(x, t)$ and $U(\xi)$ using the variable

$$\xi = x - ct.$$

2.1. Generalization of the tanh method

Assuming the integration constants vanish, we iteratively integrate this ode until the desired order is achieved, say until

$$\int \cdots \int P\Big(U, \mathrm{d}_\xi U, \mathrm{d}_\xi^2 U, \mathrm{d}_\xi^3 U, \ldots; Y\Big) = 0,$$

as long as all terms retain derivatives. We then compute for the higher-order derivatives

$$\mathrm{d}_\xi, \mathrm{d}_\xi^2, \mathrm{d}_\xi^3, \ldots, \mathrm{d}_\xi^n$$

with the highest order n present in the integrated ode. Note that this computation is particularly cumbersome.

2.1. Generalization of the tanh method

Next, we assume that the series

$$U = S(Y) = \sum_{k=0}^{M} a_k Y^k,$$

remains admissible as a solution under this generalized tanh method, allowing

$$u(x,t) = U(\xi) = S(Y)$$

to also be a solution to the ode. We balance the highest order nonlinear term with the highest order derivative following the mappings

$$u \to M, \quad u^2 \to 2M, \quad \dots, \quad u^n \to nM;$$

2.1. Generalization of the tanh method

$$\partial u \to M+1, \quad \partial^2 u \to M+2, \quad \ldots, \quad \partial^r u \to M+r.$$

We employ this to balance the highest order nonlinear term with the highest order derivative in the integrated ode and determine the balance constant M to use in the series.

We then substitute the computed derivatives and the series with the determined M into the integrated ode, grouping terms according to their powers in Y. For terms with non-integral powers of Y, we introduce forcing functions $F(Y)$ to eliminate them resulting in

$$\int \cdots \int P\left(U, \mathrm{d}_\xi U, \mathrm{d}_\xi^2 U, \mathrm{d}_\xi^3 U, \ldots; Y\right) = F(Y).$$

2.1. Generalization of the tanh method

This transforms our ode, and by extension the pde, into a forced version. To be consistent for all values of Y, the coefficient expressions must each equate to zero. This results in a nonlinear system of algebraic equations for the mathematical coefficients a_n for $n \geq 0$, $n \in \mathbb{Z}$ and physical coefficients such as the wave number μ which we'll solve.

Finally, we substitute the determined solutions for the coefficients and parameters back into the integrated ode, apply restricting conditions where necessary, and obtain a set of tunable soliton and plane periodic solutions.

2.1. Generalization of the tanh method

In summary...

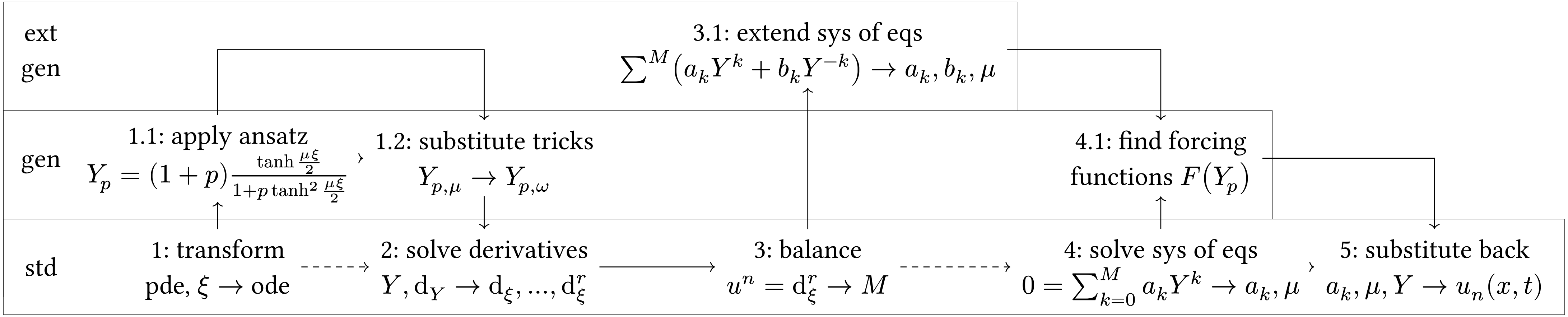


Figure 2: Procedures of the standard (std) tanh method [11,12], along with generalization (gen) [10,13–15], and subsequent novel extension (ext gen) of the generalization.

2.2. Extension of the gen tanh method

In the previous method, we only have algebraic terms in positive powers of Y in the finite series expansion above, which restricted the solution space to tanh and sech-based solutions. To explore a broader set of solutions, particularly those based on coth and csch, we extend the series to

$$S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k},$$

as inspired by an extension of the tanh method presented in [16,17]. To the best of our knowledge, this specific method has not been previously reported in the literature.

2.3. Application to the Boussinesq equation

After formulating our proposed generalization of the tanh-function method along with its extension, we implement both methods to obtain new tunable solutions to the classical form of the Boussinesq equation with $\alpha = 3$ and $\beta = 1$ [18–20]

$$\partial_t^2 u - c^2 \partial_x^2 u - \alpha \partial_x^2 u^2 - \beta \partial_x^4 u = 0.$$

3. Results and discussion

3.1. Solutions via std tanh method

Solutions to the pde

For $c^2 > 1$ (supercritical wave speed)

- y_0 gives trivial solutions
- y_1 and y_2 give the soliton solutions

$$u_1(x,t)_{\text{std}} = \frac{c^2-1}{2}\left[1-\tanh^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

$$= \frac{c^2-1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)$$

3.1. Solutions via std tanh method

$$u_2(x,t)_{\text{std}} = -\frac{c^2-1}{6}\left[1 - 3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right].$$

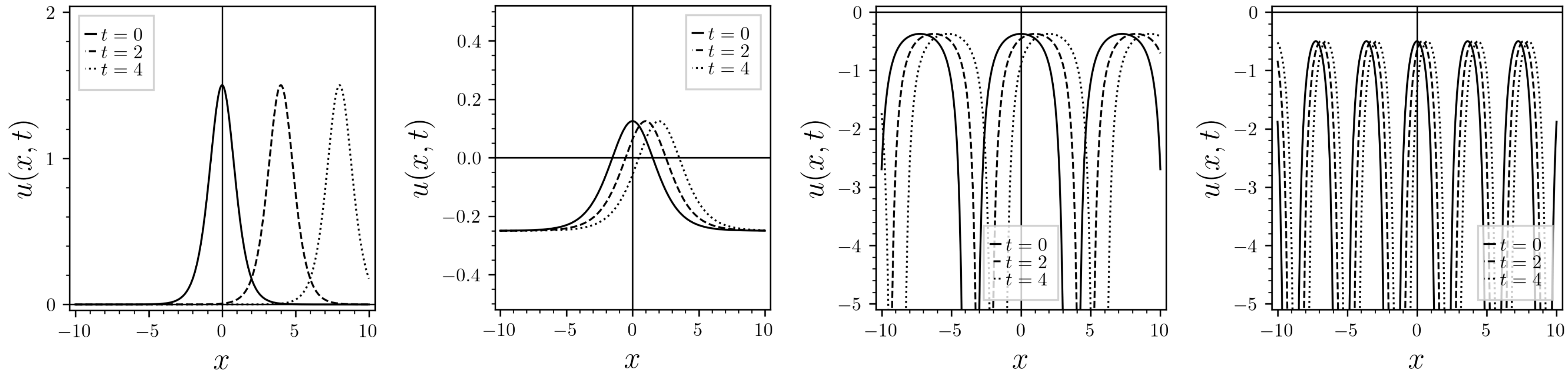
- In the opposite regime, where $c^2 < 1$ (subcritical wave speed)
  - y_1, y_2 give plane periodic solutions

$$u_3(x,t)_{\text{std}} = \frac{c^2-1}{2}\left[1 + \tan^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]$$

$$= \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

3.1. Solutions via std tanh method

$$u_4(x,t)_{\text{std}} = -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right].$$



(a): $u_{1,\text{std}} : c = 2, |x| \leq 10$ (b): $u_{2,\text{std}} : c = \frac{1}{2}, |x| \leq 10$ (c): $u_{3,\text{std}} : c = 2, |x| \leq 10$ (d): $u_{4,\text{std}} : c = \frac{1}{2}, |x| \leq 10$

Figure 3: Plots of the solutions to the classical Boussinesq equation via standard tanh method, with $t = 0, 2, 4$.

3.1. Solutions via std tanh method

Findings

- We note that u_1, u_2, u_3, u_4 correspond to the solutions found in [21]. Solutions u_1 and u_2 are the classic bell-shaped solitons, u_3 and u_4 represent a train of periodic waves with singularities
- The standard tanh method, while effective for finding these fundamental solutions, is limited because the solution forms are fixed once the balance coefficient M is determined. It does not offer inherent tunability beyond the wave speed c.

3.2. Solutions via ext tanh method

Solutions to the pde

For $c^2 > 1$ (supercritical case)

- y_0 yields trivial solutions
- y_1 and y_2 yield the soliton solutions

$$u_1(x,t)_{\text{ext std}} = \frac{c^2-1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)$$

$$u_2(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1-3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right],$$

3.2. Solutions via ext tanh method

- y_3 and y_4 yielded the non-soliton traveling wave solutions

$$u_3(x,t)_{\text{ext std}} = -\frac{c^2-1}{2}\operatorname{csch}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)$$

$$u_4(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1-3\coth^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right].$$

3.2. Solutions via ext tanh method

- y_5 and y_6 initially appear to produce distinct solutions but do not represent new 2-soliton solutions, in fact equivalent to previous ones

$$\begin{aligned} u_5(x,t)_{\text{ext std}} &= -\frac{c^2-1}{8}\left[\coth^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)\right. \\ &\qquad \left. + \tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right) - 2\right] \\ &= -\frac{c^2-1}{2}\operatorname{csch}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right) \\ &= u_3(x,t)_{\text{ext std}} \end{aligned}$$

3.2. Solutions via ext tanh method

$$u_6(x,t)_{\text{ext std}} = \frac{c^2-1}{24}\left[3\coth^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)\right.$$
$$\left. +3\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)+2\right]$$
$$= u_4(x,t)_{\text{ext std}}.$$

3.2. Solutions via ext tanh method

For the opposite regime where $c^2 < 1$ (subcritical case)

- y_1, y_2, y_3 and y_4 give the plane periodic solutions

$$u_7(x,t)_{\text{ext std}} = \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

$$u_8(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

$$u_9(x,t)_{\text{ext std}} = \frac{c^2-1}{2}\csc^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

3.2. Solutions via ext tanh method

$$u_{10}(x,t)_{\text{ext std}} = -\frac{c^2-1}{6}\left[1+3\cot^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

- Again, y_5 and y_6 in this regime lead to non-unique solutions because

$$u_{11}(x,t)_{\text{ext std}} = \frac{c^2-1}{8}\left[\cot^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)\right.$$
$$\left.+\tan^2\left(\frac{\sqrt{1-c^2}}{4}\right)+2\right]$$

3.2. Solutions via ext tanh method

$$= \frac{c^2 - 1}{2} \csc^2\left(\frac{\sqrt{1 - c^2}}{2}(x - ct)\right)$$

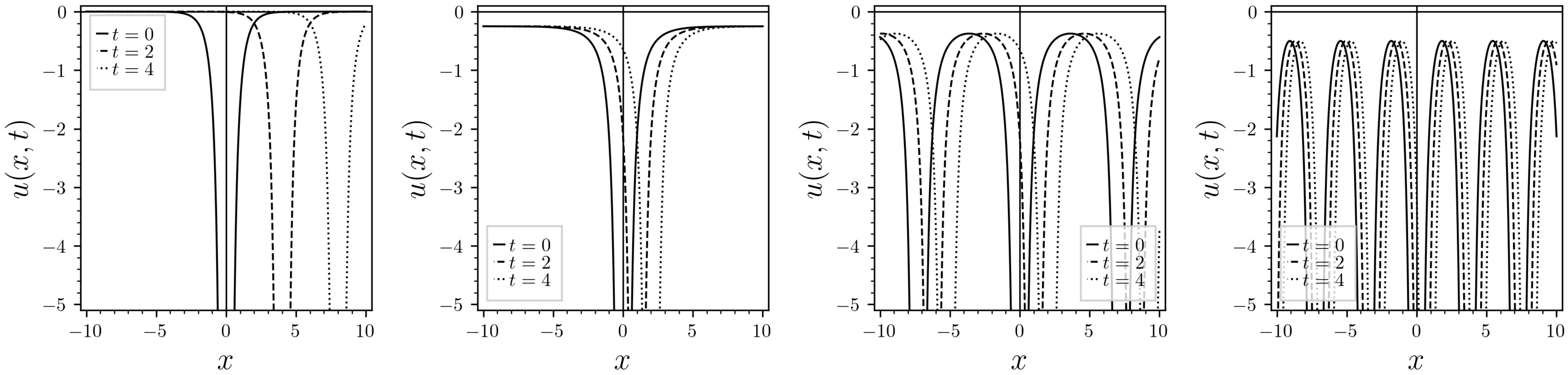
$$= u_9(x, t)_{\text{ext std}}$$

$$u_{12}(x, t)_{\text{ext std}} = -\frac{c^2 - 1}{24}\left[3\cot^2\left(\frac{\sqrt{c^2 - 1}}{4}(x - ct)\right)\right.$$

$$\left. + 3\tan^2\left(\frac{\sqrt{c^2 - 1}}{4}(x - ct)\right) - 2\right]$$

$$= u_{10}(x, t)_{\text{ext std}}.$$

3.2. Solutions via ext tanh method



(a): $u_{3,\text{ext std}} : c = 2, |x| \leq 10$ (b): $u_{4,\text{ext std}} : c = \frac{1}{2}, |x| \leq 10$ (c): $u_{9,\text{ext std}} : c = 2, |x| \leq 10$ (d): $u_{10,\text{ext std}} : c = \frac{1}{2}, |x| \leq 10$

Figure 4: Plots of the additional solutions to the classical Boussinesq equation via extended standard tanh method, with $t = 0, 2, 4$. The other solutions are found in Figure 3.

3.2. Solutions via ext tanh method

Findings

- Solutions u_3 and u_9 involving csch and csc functions represent waves with singularities, with Y^{-k} terms effectively doubling obtainable solution forms
- Extended method uncovered richer variety of exact solutions consistent with existing literature, though some coefficient combinations produced redundant solutions

3.3. Computing the derivatives d_ξ, d_ξ^2

The core of our generalization lies in the novel ansatz

$$Y_{p,\xi} = Y_p(\mu\xi) = (1+p)\frac{\tanh\dfrac{\mu\xi}{2}}{1+p\tanh^2\dfrac{\mu\xi}{2}}, \qquad 0 \le p \le 1, \quad p \in \mathbb{R}.$$

introduced as a new independent variable where $\xi = x - ct$ and μ is wave number. To substitute this ansatz into the ode derived from the Boussinesq equation, we expressed its derivatives $\mathrm{d}_\xi Y_p$ and $\mathrm{d}_\xi^2 Y_p$ in terms of Y_p itself. This subsection details this crucial mathematical step.

The derivation involved an auxiliary variable transformation

3.3. Computing the derivatives d_ξ, d_ξ^2

$$\tanh\frac{\mu\xi}{2} = \frac{1}{\sqrt{p}}\tanh\frac{\mu\omega}{2}$$

$$\begin{aligned}
\Longrightarrow Y_{p,\xi} &= (1+p)\frac{\frac{1}{\sqrt{p}}\tanh\frac{\mu\omega}{2}}{1+p\frac{1}{p}\tanh^2\frac{\mu\omega}{2}} \\
&= \frac{p+1}{2\sqrt{p}}\frac{2\tanh\frac{\mu\omega}{2}}{1+\tanh^2\frac{\mu\omega}{2}} \\
&= \frac{p+1}{2\sqrt{p}}\tanh\mu\omega \\
&= Y_p(\omega) \equiv Y_{p,\omega}.
\end{aligned}$$

3.3. Computing the derivatives d_ξ, d_ξ^2

Note that $\mathrm{d}_\xi = \mathrm{d}_\omega Y_p \cdot \mathrm{d}_{Y_p} = \mathrm{d}_\xi \omega \cdot \mathrm{d}_\omega Y_p \cdot \mathrm{d}_{Y_p}$ and $\omega = \frac{2}{\mu} \operatorname{arctanh}\left(\sqrt{p} \tanh \frac{\mu\xi}{2}\right)$. Applying the chain rule, we first computed

$$
\begin{aligned}
\mathrm{d}_\omega Y_p &= \mathrm{d}_\omega Y_{p,\omega} \\
&= \frac{p+1}{2\sqrt{p}} \mu (1 - \tanh^2 \mu\omega) \\
&= \frac{p+1}{2\sqrt{p}} \mu \left[1 - \left(\frac{2\sqrt{p}}{p+1} \right)^2 \left(\frac{p+1}{2\sqrt{p}} \right)^2 \tanh^2 \mu\omega \right] \\
&= \frac{p+1}{2\sqrt{p}} \mu \left[1 - \left(\frac{2\sqrt{p}}{p+1} \right)^2 Y_{p,\omega}^2 \right]
\end{aligned}
$$

3.3. Computing the derivatives d_ξ, d_ξ^2

$$= \frac{p+1}{2\sqrt{p}}\left(\frac{2\sqrt{p}}{p+1}\right)^2 \mu\left[\left(\frac{p+1}{2\sqrt{p}}\right)^2 - Y_{p,\omega}^2\right]$$

$$= \frac{\mu}{q_p}\left(q_p^2 - Y_{p,\omega}^2\right)$$

where $q_p \equiv \frac{p+1}{2\sqrt{p}}$. Next, we computed

$$\mathrm{d}_\xi \omega = \frac{1}{\sqrt{p}} \frac{p - \tanh^2 \frac{\mu\omega}{2}}{1 - \tanh^2 \frac{\mu\omega}{2}}$$

3.3. Computing the derivatives d_ξ, d_ξ^2

$$= \frac{1}{\sqrt{p}} \frac{p+1}{2} \left[(1-1) + \frac{2p - 2\tanh^2 \frac{\mu\omega}{2}}{(p+1)(1-\tanh^2 \frac{\mu\omega}{2})} \right]$$

$$= \frac{1}{\sqrt{p}} \frac{p+1}{2} \left[1 + \frac{-(p-1)(1-\tanh^2 \frac{\mu\omega}{2}) + 2p - 2\tanh^2 \frac{\mu\omega}{2}}{(p+1)(1-\tanh^2 \frac{\mu\omega}{2})} \right]$$

$$= \frac{1}{\sqrt{p}} \frac{p+1}{2} \left[1 + \frac{(p-1)(1+\tanh^2 \frac{\mu\omega}{2})}{(p+1)(1-\tanh^2 \frac{\mu\omega}{2})} \right]$$

$$= \frac{1}{\sqrt{p}} \frac{p+1}{2} \left\{ 1 + \frac{p-1}{p+1} \left[\left(\frac{1-\tanh^2 \frac{\mu\omega}{2}}{1+\tanh^2 \frac{\mu\omega}{2}} \right)^2 \right]^{1/2} \right\}.$$

3.3. Computing the derivatives d_ξ, d_ξ^2

To simplify the innermost term, we have

$$\left(\frac{1-\tanh^2\frac{\mu\omega}{2}}{1+\tanh^2\frac{\mu\omega}{2}}\right)^2 = \frac{1-2\tanh^2\frac{\mu\omega}{2}+\tanh^4\frac{\mu\omega}{2}}{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2}$$

$$= \frac{1+2\tanh^2\frac{\mu\omega}{2}+\tanh^4\frac{\mu\omega}{2}-4\tanh^2\frac{\mu\omega}{2}}{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2}$$

3.3. Computing the derivatives d_ξ, d_ξ^2

$$= \frac{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2 - 4\tanh^2\frac{\mu\omega}{2}}{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2}$$

$$= 1 - \frac{4\tanh^2\frac{\mu\omega}{2}}{\left(1+\tanh^2\frac{\mu\omega}{2}\right)^2}$$

$$= 1 - \tanh^2\mu\omega$$

$$= \left(\frac{2\sqrt{p}}{p+1}\right)^2 \left[\left(\frac{p+1}{2\sqrt{p}}\right)^2 - \left(\frac{p+1}{2\sqrt{p}}\right)^2 \tanh^2\mu\omega\right]$$

3.3. Computing the derivatives d_ξ, d_ξ^2

$$= \frac{1}{q_p^2}\left(q_p^2 - Y_{p,\xi}^2\right).$$

With $r_p \equiv \frac{p-1}{2\sqrt{p}} = \frac{p-1}{p+1} q_p$, we obtained

$$\mathrm{d}_\xi \omega = q_p\left[1 + \frac{p-1}{p+1} q_p \left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right]$$

$$= q_p\left[1 + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right].$$

Finally, the resulting expressions for the first and second derivatives were

3.3. Computing the derivatives d_ξ, d_ξ^2

$$\begin{aligned}\mathrm{d}_\xi &= \mathrm{d}_\xi \omega \cdot \mathrm{d}_\omega Y_p \cdot d_{Y_p} \\ &= q_p \left[1 + r_p \left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right] \frac{\mu}{q_p} \left(q_p^2 - Y_{p,\xi}^2\right) \mathrm{d}_{Y_p} \\ &= \mu \left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p \left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right] \mathrm{d}_{Y_p}\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}_\xi^2 &= \mathrm{d}_\xi \left\{\mu \left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p \left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right] \mathrm{d}_{Y_p}\right\} \\ &= \mu \left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p \left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]\end{aligned}$$

3.3. Computing the derivatives d_ξ, d_ξ^2

$$d_{Y_p}\left\{\mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]d_{Y_p}\right\}$$

$$= \mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]$$

$$\left\{\mu\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]d_{Y_p}^2\right.$$

$$\left.+\mu\left[\left(0 - 2Y_{p,\xi}\right) + r_p\frac{1}{2}\left(0 - 2Y_{p,\xi}\right)\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right]d_{Y_p}\right\}$$

$$= \mu^2\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]^2 d_{Y_p}^2$$

3.3. Computing the derivatives d_ξ, d_ξ^2

$$+\mu^2\left[\left(q_p^2 - Y_{p,\xi}^2\right) + r_p\left(q_p^2 - Y_{p,\xi}^2\right)^{1/2}\right]$$

$$\left[-2Y_{p,\xi} - r_p Y_{p,\xi}\left(q_p^2 - Y_{p,\xi}^2\right)^{-1/2}\right]\mathrm{d}_{Y_p}.$$

- This computation provides an improvement in conciseness over previous the operational rules for how derivatives of $u(x,t)$, expressed as a series in Y_p, transform. It is also an improvement in conciseness compared to previous work [13]. The complexity of these derivatives, particularly due to $\left(q_p^2 - Y_p^2\right)^{\frac{1}{2}}$, highlights the algebraic intricacy of the generalized method. This explains the necessity of introducing a forcing function $F\left(Y_p\right)$ when $p \neq 1$

3.4. Solutions via gen tanh method

Terms involving non-integer powers of Y_p, specifically those with $\left(q_p^2 - Y_p^2\right)^{\pm\frac{1}{2}}$, emerged. These terms cannot be balanced by integer powers of Y_p alone. To address this, we introduce a forcing function

$$
\begin{aligned}
F(Y) = {} & 2a_2\mu^2q^2rY_p^2\left(q_p^2 - Y_p^2\right)^{-1/2} + a_1\mu^2rY_p^3\left(q_p^2 - Y_p^2\right)^{-1/2} \\
& -2a_2\mu^2rY_p^4\left(q_p^2 - Y_p^2\right)^{-1/2} - 4a_2\mu^2q^2r\left(q_p^2 - Y_p^2\right)^{1/2} \\
& +2a_1\mu^2rY_p\left(q_p^2 - Y_p^2\right)^{1/2} + a_1\mu^2q^2rY_p^2\left(q_p^2 - Y_p^2\right)^{1/2} \\
= {} & \left[2a_2\mu^2q^2rY_p^2 + a_1\mu^2rY_p^3 - 2a_2\mu^2rY_p^4\right]\left(q_p^2 - Y_p^2\right)^{-1/2} \\
& +\left[-4a_2\mu^2q^2r + 2a_1\mu^2rY_p + a_1\mu^2q^2rY_p^2\right]\left(q_p^2 - Y_p^2\right)^{1/2}
\end{aligned}
$$

which we note can be further simplified. By equating the original nonlinear ode, and consequently the nonlinear pde, to this forcing function, we obtain a forced version of the Boussinesq equation

$$
\begin{aligned}
&(c^2 - 1)u - 3u^2 - \mathrm{d}_\xi^2 u = F(Y) \\
&\implies \partial_t^2 u - \partial_x^2 u - \partial_x^2 (3u^2) - \partial_x^4 u = F(Y).
\end{aligned}
$$

This modification allowed us to eliminate terms with non-integral powers of Y. Importantly, the original, unforced Boussinesq equation is recovered by setting $p = 1$, which makes $r_p = 0$ therefore $F(Y_p) = 0$.

3.4. Solutions via gen tanh method

Solutions to the pde, when $c^2 > 1$

- y_0 gives trivial solutions
- y_1 and y_2 provide soliton solutions

$$u_1(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

$$+(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

3.4. Solutions via gen tanh method

$$u_2(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}\left(1 - \frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

$$+(c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2.$$

3.4. Solutions via gen tanh method

In the opposite regime where $c^2 < 1$

- y_1 and y_2 yield the plane periodic solutions

$$u_3(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

$$+(c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2$$

3.4. Solutions via gen tanh method

$$u_4(x,t,p)_{\text{gen}} = \frac{c^2-1}{6}\left(1 - \frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$
$$+(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}$$
$$\left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

3.4. Solutions via gen tanh method

Quick verification: setting $p = 1$ in these generalized solutions produces the unforced particular solutions obtained via the standard tanh method

$$u_1(x,t,p=1)_{\text{gen}} = \frac{c^2-1}{2} + 2(1-c^2)\left[\frac{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

$$= \frac{c^2-1}{2}\left[1-\tanh^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

$$= \frac{c^2-1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)$$

3.4. Solutions via gen tanh method

$$= u_1(x,t)_{\text{ext std}} = u_1(x,t)_{\text{std}}$$

$$u_2(x,t,p=1)_{\text{gen}} = -\frac{c^2-1}{6} + 2(c^2-1)\left[\frac{\tanh\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2$$

$$= -\frac{c^2-1}{6}\left[1-3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]$$

$$= u_2(x,t)_{\text{ext std}} = u_2(x,t)_{\text{std}}$$

3.4. Solutions via gen tanh method

$$u_3(x,t,p=1)_{\text{gen}} = \frac{c^2-1}{2} + 2(c^2-1)\left[\frac{\tan\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2$$

$$= \frac{c^2-1}{2}\left[1+\tan^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]$$

$$= \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

$$= u_7(x,t)_{\text{ext std}} = u_3(x,t)_{\text{std}}$$

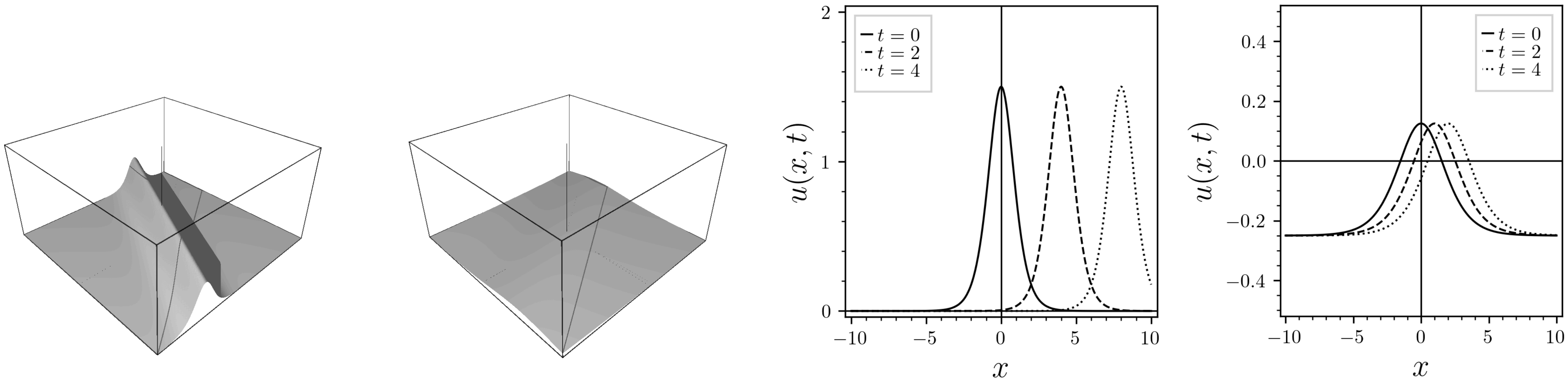
3.4. Solutions via gen tanh method

$$u_4(x,t,p=1)_{\text{gen}} = -\frac{c^2-1}{6} + 2(1-c^2)\left[\frac{\tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

$$= -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

$$= u_8(x,t)_{\text{ext std}} = u_4(x,t)_{\text{std}}$$

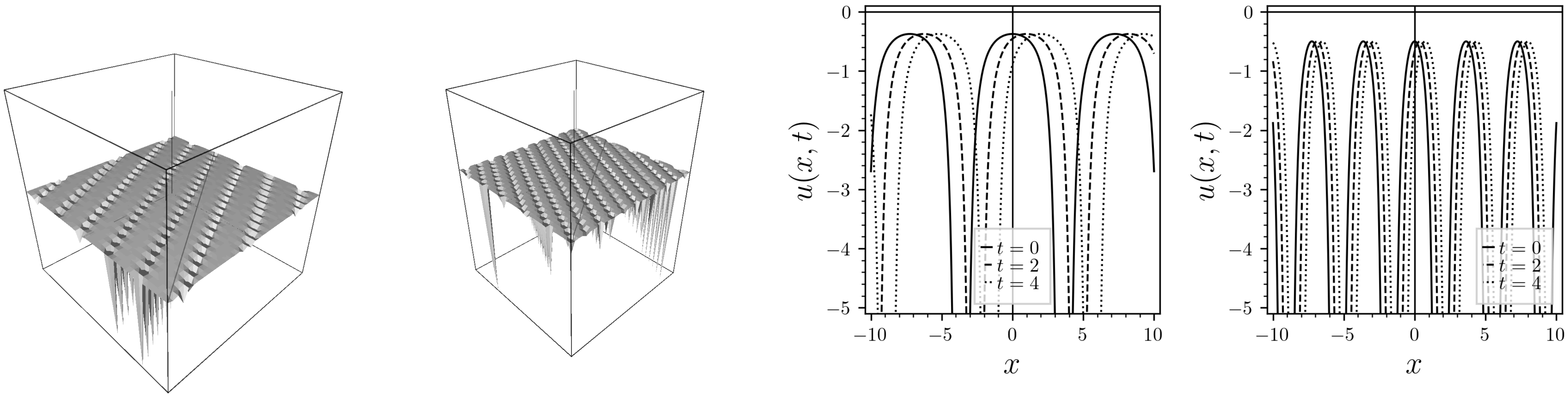
3.4. Solutions via gen tanh method



(a): $u_{1,\text{gen}} : c = 2, |x,t| \le 4$ (b): $u_{2,\text{gen}} : c = \frac{1}{2}, |x,t| \le 4$ (c): $u_{1,\text{gen}} : c = 2, |x| \le 10$ (d): $u_{2,\text{gen}} : c = \frac{1}{2}, |x| \le 10$

Figure 5: Plots of the soliton solutions to the classical Boussinesq equation via generalized tanh method, with $t = 0, 2, 4$.

3.4. Solutions via gen tanh method



(a): $u_{3,\text{gen}} : c = 2, |x, t| \leq 4$ (b): $u_{4,\text{gen}} : c = \frac{1}{2}, |x, t| \leq 4$ (c): $u_{3,\text{gen}} : c = 2, |x| \leq 10$ (d): $u_{4,\text{gen}} : c = \frac{1}{2}, |x| \leq 10$

Figure 6: Plots of the plane periodic solutions to the classical Boussinesq equation via generalized tanh method, with $t = 0, 2, 4$.

3.4. Solutions via gen tanh method

Findings

- The generalized tanh method, for $p \neq 1$, yields solutions to the forced Boussinesq equation due to the emergence of terms involving non-integer powers of Y_p, represented by the forcing function $F(Y_p)$. Setting $p = 1$ eliminates $F(Y_p)$, resulting in solutions to the original, unforced Boussinesq equation.
- This approach introduces a valuable parameter p, which provides a continuous deformation of the standard solutions while revealing solutions to related forced systems

3.5. Solutions via ext gen tanh method

The terms involving non-integer powers of Y_p already separated into the forcing function

$$
\begin{aligned}
F(Y) = {} & -2b_2\mu^2q^2rY_p^{-2}\left(q_p^2 - Y_p^2\right)^{-1/2} - b_1\mu^2q^2rY_p^{-1}\left(q_p^2 - Y_p^2\right)^{-1/2} \\
& +2b_2\mu^2r\left(q_p^2 - Y_p^2\right)^{-1/2} + \left(a_1\mu^2q^2r + b_1\mu^2r\right)Y_p\left(q_p^2 - Y_p^2\right)^{-1/2} \\
& +2a_2\mu^2q^2rY_p^2\left(q_p^2 - Y_p^2\right)^{-1/2} - a_1\mu^2rY_p^3\left(q_p^2 - Y_p^2\right)^{-1/2} \\
& -2a_2\mu^2rY_p^4\left(q_p^2 - Y_p^2\right)^{-1/2} - 12b_2\mu^2q^2rY_p^{-4}\left(q_p^2 - Y_p^2\right)^{1/2} \\
& -4b_1\mu^2q^2rY_p^{-3}\left(q_p^2 - Y_p^2\right)^{1/2} + 8b_2\mu^2rY_p^{-2}\left(q_p^2 - Y_p^2\right)^{1/2} \\
& +2b_1\mu^2rY_p^{-1}\left(q_p^2 - Y_p^2\right)^{1/2} + \left(-4a_2\mu^2q^2r - b_1\mu^2q^2r\right)\left(q_p^2 - Y_p^2
\end{aligned}
$$

3.5. Solutions via ext gen tanh method

$$
\begin{aligned}
&+2a_1\mu^2 rY_p\left(q_p^2 - Y_p^2\right)^{1/2} + 8a_2\mu^2 rY_p^2\left(q_p^2 - Y_p^2\right)^{1/2} \\
&= \left[-2b_2\mu^2 q^2 rY_p^{-2} - b_1\mu^2 q^2 rY_p^{-1} + 2b_2\mu^2 r + \left(a_1\mu^2 q^2 r + b_1\mu^2 r\right)Y \right.\\
&\left. +2a_2\mu^2 q^2 rY_p^2 - a_1\mu^2 rY_p^3 - 2a_2\mu^2 rY_p^4\right]\left(q_p^2 - Y_p^2\right)^{-1/2} \\
&+\left[-12b_2\mu^2 q^2 rY_p^{-4} - 4b_1\mu^2 q^2 rY_p^{-3} + 8b_2\mu^2 rY_p^{-2} + 2b_1\mu^2 rY_p \right.\\
&\left. +\left(-4a_2\mu^2 q^2 r - b_1\mu^2 q^2 r\right) + 2a_1\mu^2 rY_p + 8a_2\mu^2 rY_p^2\right] \\
&\left(q_p^2 - Y_p^2\right)^{1/2}.
\end{aligned}
$$

- By equating our nonlinear ode to $F(Y_p)$, the equation becomes forced. But by setting $p = 1$, we recover the unforced system

3.5. Solutions via ext gen tanh method

Solutions to the pde, when $c^2 > 1$

- y_0 provides trivial solutions
- y_1 and y_2 give soliton solutions

$$u_1(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

$$+(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

3.5. Solutions via ext gen tanh method

$$u_2(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1 - \frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right) + (c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}} \left[\frac{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2$$

3.5. Solutions via ext gen tanh method

- y_3 and y_4 give the non-soliton traveling wave solutions

$$u_3(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

$$+(1-c^2)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

3.5. Solutions via ext gen tanh method

$$u_4(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1 - \frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

$$+(c^2-1)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{1+p\tanh^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{\tanh\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2$$

3.5. Solutions via ext gen tanh method

In the opposite regime where $c^2 < 1$

- y_1, y_2, y_3 and y_4 give plane periodic solutions

$$u_5(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1+\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right) + (c^2-1)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}} \left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2$$

3.5. Solutions via ext gen tanh method

$$u_6(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1 - \frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$
$$+(1-c^2)\frac{2p(p+1)^2}{\sqrt{(3p^2+1)(p^2+3)}}$$
$$\left[\frac{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

$$u_7(x,t,p)_{\text{ext gen}} = \frac{c^2-1}{6}\left(1 + \frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

3.5. Solutions via ext gen tanh method

$$+(c^2-1)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{1-p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{1-c^2}}{2}(x-ct)\right)}\right]^2$$

$$u_8(x,t,p)_{\text{ext gen}}=\frac{c^2-1}{6}\left(1-\frac{3p^2+2p+3}{\sqrt{(3p^2+1)(p^2+3)}}\right)$$

$$+(1-c^2)\frac{p^2+1}{4p\sqrt{(3p^2+1)(p^2+3)}}$$

$$\left[\frac{1 - p\tan^2\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}{\tan\left(\frac{\sqrt{p}}{\sqrt[4]{(3p^2+1)(p^2+3)}}\frac{\sqrt{c^2-1}}{2}(x-ct)\right)}\right]^2$$

Quick check: setting $p = 1$. This should reduce the forced generalized extended solutions to the unforced extended standard solutions

$$u_1(x,t,p=1)_{\text{ext gen}} = \frac{c^2-1}{2} + 2(1-c^2)\left[\frac{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

3.5. Solutions via ext gen tanh method

$$= \frac{c^2 - 1}{2}\left[1 - \tanh^2\left(\frac{\sqrt{c^2-1}}{2}(x - ct)\right)\right]$$

$$= \frac{c^2 - 1}{2}\operatorname{sech}^2\left(\frac{\sqrt{c^2-1}}{2}(x - ct)\right)$$

$$= u_1(x,t)_{\text{gen}} = u_1(x,t)_{\text{ext std}} = u_1(x,t)_{\text{std}}$$

$$u_2(x,t,p=1)_{\text{ext gen}} = -\frac{c^2-1}{6} + 2(c^2-1)\left[\frac{\tanh\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1+\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right.$$

$$= -\frac{c^2-1}{6}\left[1 - 3\tanh^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]$$

$$= u_2(x,t)_{\text{gen}} = u_2(x,t)_{\text{ext std}} = u_2(x,t)_{\text{std}}$$

$$u_3(x,t,p=1)_{\text{ext gen}} = \frac{c^2-1}{2} + \frac{1-c^2}{8}\left[\frac{1+\tanh^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{\tanh\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

$$= \frac{c^2-1}{2}\left[1 - \coth^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

3.5. Solutions via ext gen tanh method

$$= -\frac{c^2-1}{2}\operatorname{csch}^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)$$

$$= u_3(x,t)_{\text{ext std}}$$

$$u_4(x,t,p=1)_{\text{ext gen}} = -\frac{c^2-1}{6} + \frac{c^2-1}{8}\left[\frac{1+\tanh^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{\tanh\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2$$

$$= -\frac{c^2-1}{6}\left[1-3\coth^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]$$

$$= u_4(x,t)_{\text{ext std}}$$

3.5. Solutions via ext gen tanh method

$$
\begin{aligned}
u_5(x,t,p=1)_{\text{ext gen}} &= \frac{c^2-1}{2} + 2(c^2-1)\left[\frac{\tan\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2 \\
&= \frac{c^2-1}{2}\left[1+\tan^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right] \\
&= \frac{c^2-1}{2}\sec^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right) \\
&= u_3(x,t)_{\text{gen}} = u_7(x,t)_{\text{ext std}} = u_3(x,t)_{\text{std}}
\end{aligned}
$$

3.5. Solutions via ext gen tanh method

$$u_6(x,t,p=1)_{\text{ext gen}} = -\frac{c^2-1}{6} + 2(1-c^2)\left[\frac{\tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{1-\tan^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

$$= -\frac{c^2-1}{6}\left[1+3\tan^2\left(\frac{\sqrt{c^2-1}}{2}(x-ct)\right)\right]$$

$$= u_4(x,t)_{\text{gen}} = u_8(x,t)_{\text{ext std}} = u_4(x,t)_{\text{std}}$$

$$u_7(x,t,p=1)_{\text{ext gen}} = \frac{c^2-1}{2} + \frac{c^2-1}{8}\left[\frac{1-\tan^2\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}{\tan\left(\frac{\sqrt{1-c^2}}{4}(x-ct)\right)}\right]^2$$

$$= \frac{c^2-1}{2}\left[1+\cot^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)\right]$$

$$= \frac{c^2-1}{2}\csc^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right)$$

$$= u_9(x,t)_{\text{ext std}}$$

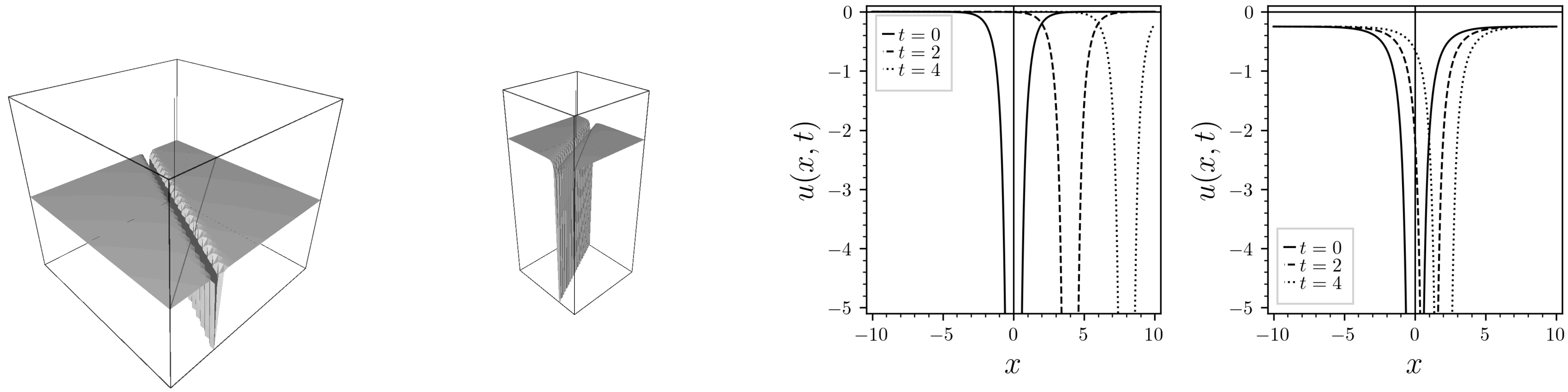
$$u_8(x,t,p=1)_{\text{ext gen}} = -\frac{c^2-1}{6}+\frac{1-c^2}{8}\left[\frac{1-\tan^2\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}{\tan\left(\frac{\sqrt{c^2-1}}{4}(x-ct)\right)}\right]^2$$

3.5. Solutions via ext gen tanh method

$$= -\frac{c^2 - 1}{6}\left[1 + 3\cot^2\left(\frac{\sqrt{c^2 - 1}}{2}(x - ct)\right)\right]$$

$$= u_{10}(x, t)_{\text{ext std}}.$$

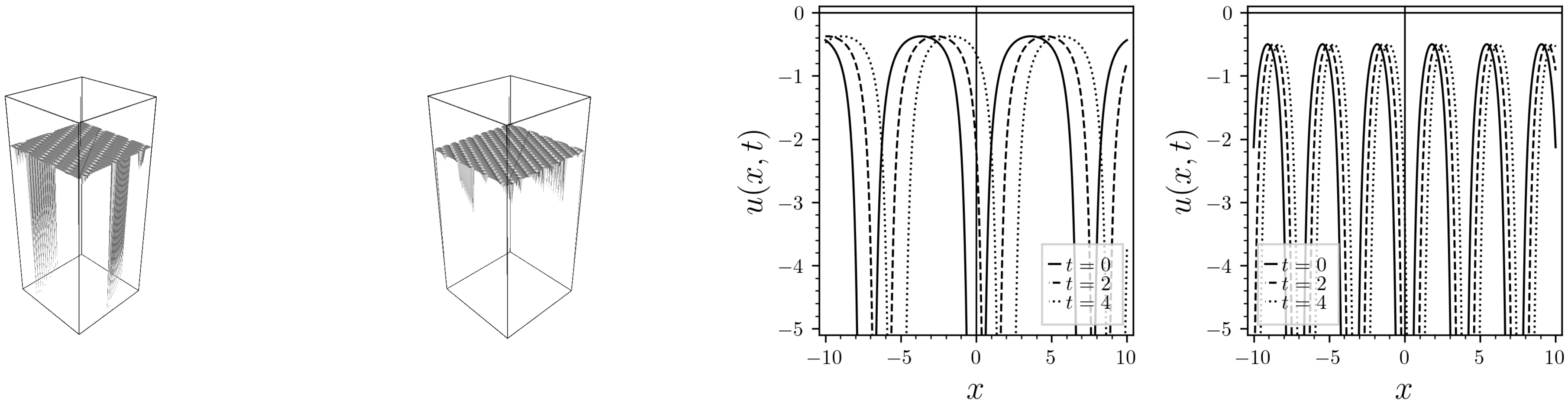
3.5. Solutions via ext gen tanh method



(a): $u_{3,\text{ext gen}} : c = 2, |x,t| \leq 4$ (b): $u_{4,\text{ext gen}} : c = \frac{1}{2}, |x,t| \leq 4$ (c): $u_{3,\text{ext gen}} : c = 2, |x| \leq 10$ (d): $u_{4,\text{ext gen}} : c = \frac{1}{2}, |x| \leq 10$

Figure 7: Plots of the additional non-soliton traveling wave solutions to the classical Boussinesq equation via extended generalized tanh method, with $t = 0, 2, 4$. The other solutions are found in Figure 5 and Figure 6.

3.5. Solutions via ext gen tanh method



(a): $u_{7,\text{ext gen}} : c = 2, |x, t| \leq 4$ (b): $u_{8,\text{ext gen}} : c = \frac{1}{2}, |x, t| \leq 4$ (c): $u_{7,\text{ext gen}} : c = 2, |x| \leq 10$ (d): $u_{8,\text{ext gen}} : c = \frac{1}{2}, |x| \leq 10$

Figure 8: Plots of the additional plane periodic solutions to the classical Boussinesq equation via extended generalized tanh method, with $t = 0, 2, 4$. The other solutions are found in Figure 5 and Figure 6.

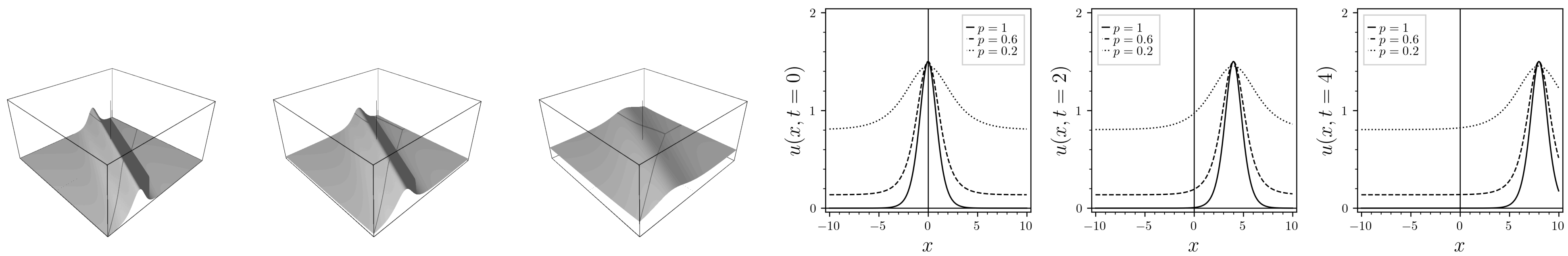
3.5. Solutions via ext gen tanh method

Findings

- Extended generalized method identified 14 solution sets, yielding 8 unique families after removing trivial and duplicate solutions—comprising 2 solitons, 2 non-soliton traveling waves, and 4 plane periodic solutions for forced Boussinesq equation
- Setting $p = 1$ correctly reduces solutions to those from extended standard tanh method, confirming that extended generalized method encompasses previous approaches while providing additional tunable families, establishing validity of ansatz-inspired function Y_p

3.6. Playing with the parameter p

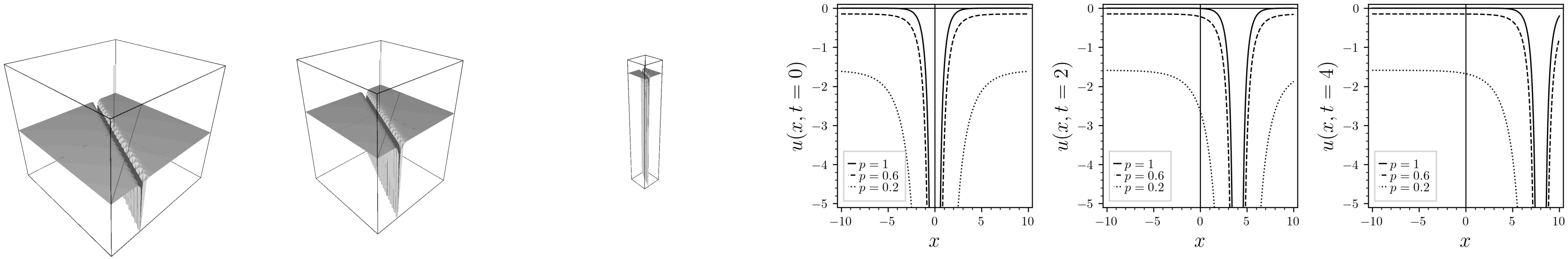
We use $u_{1,\text{ext gen}}$, $u_{3,\text{ext gen}}$, and $u_{6,\text{ext gen}}$, with $c = 2$, as representative examples of soliton, non-soliton traveling wave, and plane periodic solutions, respectively. We highlight the control we have over the solutions by using p as our tunable parameter:



(a): $u_1 : |x, t| \leq 4, p = 1.0$ (b): $u_1 : |x, t| \leq 4, p = 0.6$ (c): $u_1 : |x, t| \leq 4, p = 0.2$ (d): $u_1 : |x| \leq 10, t = 0$ (e): $u_1 : |x| \leq 10, t = 2$ (f): $u_1 : |x| \leq 10, t = 4$

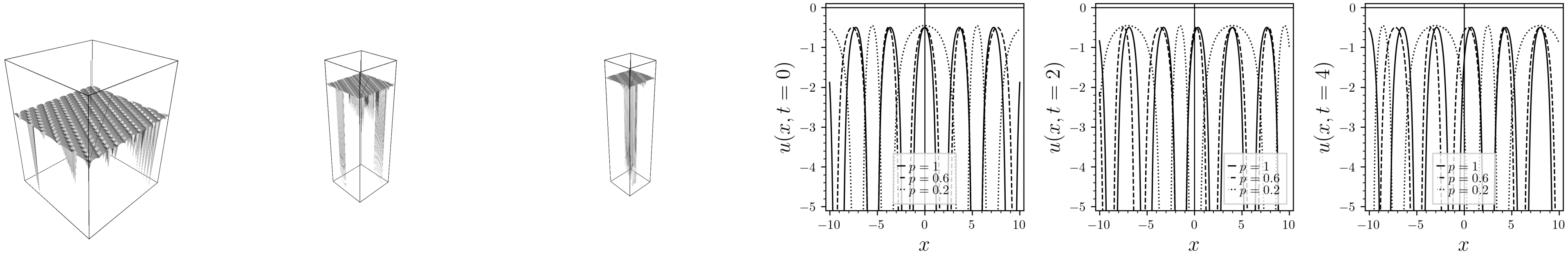
Figure 9: Spacetime evolutions of the soliton solution $u_{1,\text{ext gen}}$ for $c = 2$ and $0 \leq p \leq 1$.

3.6. Playing with the parameter p



(a): $u_3 : |x,t| \leq 15, p = 1.0$ (b): $u_3 : |x,t| \leq 15, p = 0.6$ (c): $u_3 : |x,t| \leq 15, p = 0.2$ (d): $u_3 : |x| \leq 10, t = 0$ (e): $u_3 : |x| \leq 10, t = 2$ (f): $u_3 : |x| \leq 10, t = 4$

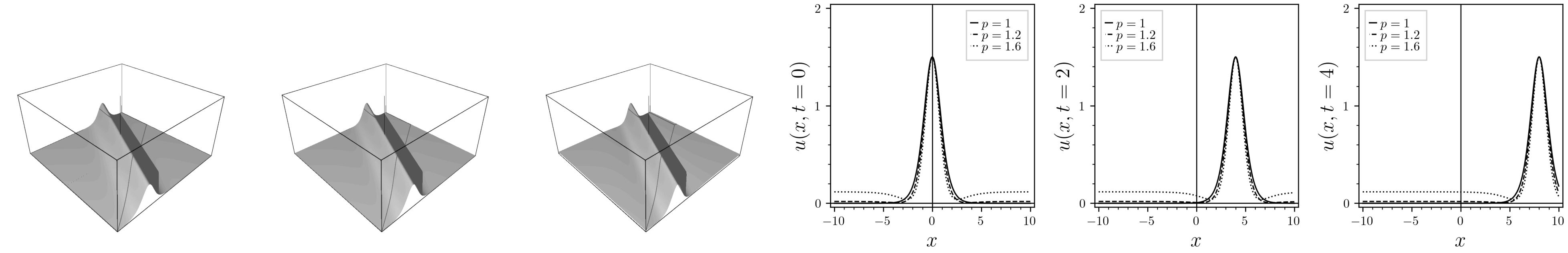
Figure 10: Spacetime evolutions of the non-soliton traveling wave solution $u_{3,\text{ext gen}}$ for $c = 2$ and $0 \leq p \leq 1$.



(a): $u_6 : |x,t| \leq 1000, p = 1.0$ (b): $u_6 : |x,t| \leq 1000, p = 0.6$ (c): $u_6 : |x,t| \leq 1000, p = 0.2$ (d): $u_6 : |x| \leq 10, t = 0$ (e): $u_6 : |x| \leq 10, t = 2$ (f): $u_6 : |x| \leq 10, t = 4$

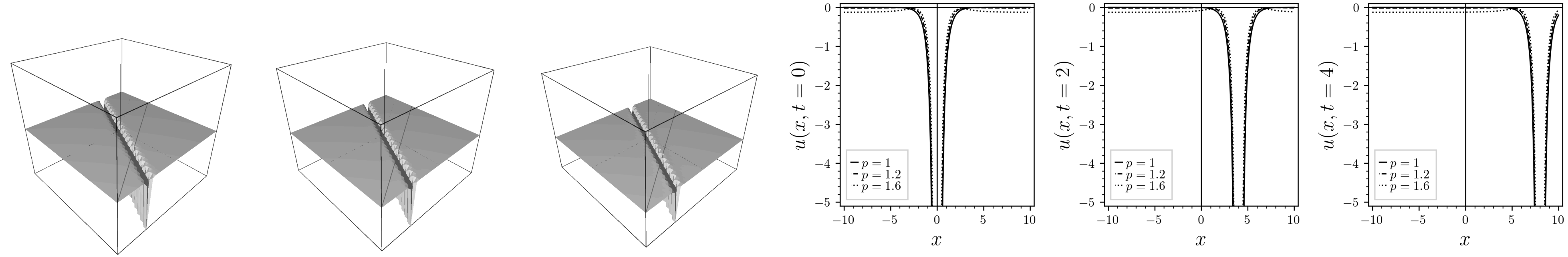
Figure 11: Spacetime evolutions of the plane periodic solution $u_{6,\text{ext gen}}$ for $c = 2$ and $0 \leq p \leq 1$.

3.6. Playing with the parameter p



(a): $u_1 : |x,t| \leq 4, p = 1.0$ (b): $u_1 : |x,t| \leq 4, p = 1.2$ (c): $u_1 : |x,t| \leq 4, p = 1.6$ (d): $u_1 : |x| \leq 10, t = 0$ (e): $u_1 : |x| \leq 10, t = 2$ (f): $u_1 : |x| \leq 10, t = 4$

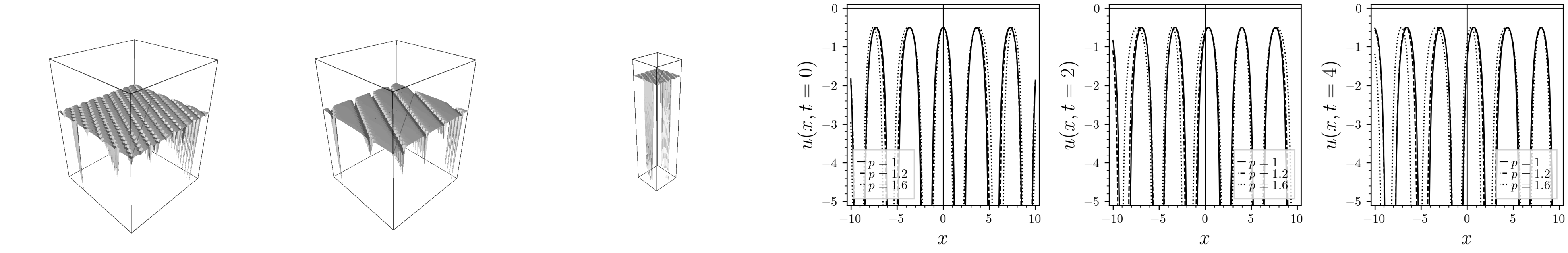
Figure 12: Spacetime evolutions of the soliton solution $u_{1,\text{ext gen}}$ for $c = 2$ and $p > 1$.



(a): $u_3 : |x,t| \leq 15, p = 1.0$ (b): $u_3 : |x,t| \leq 15, p = 1.2$ (c): $u_3 : |x,t| \leq 15, p = 1.6$ (d): $u_3 : |x| \leq 10, t = 0$ (e): $u_3 : |x| \leq 10, t = 2$ (f): $u_3 : |x| \leq 10, t = 4$

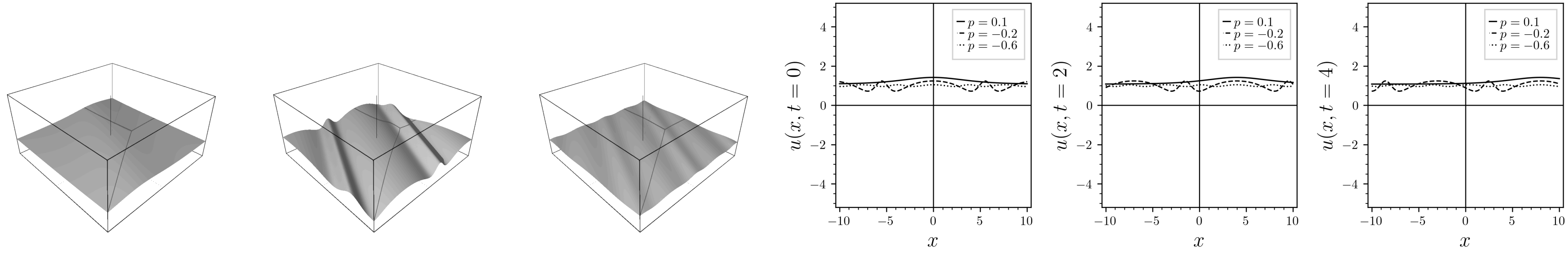
Figure 13: Spacetime evolutions of the non-soliton traveling wave solution $u_{3,\text{ext gen}}$ for $c = 2$ and $p > 1$.

3.6. Playing with the parameter p



(a): $u_6 : |x,t| \leq 1000, p = 1.0$ (b): $u_6 : |x,t| \leq 1000, p = 1.2$ (c): $u_6 : |x,t| \leq 1000, p = 1.6$ (d): $u_6 : |x| \leq 10, t = 0$ (e): $u_6 : |x| \leq 10, t = 2$ (f): $u_6 : |x| \leq 10, t = 4$

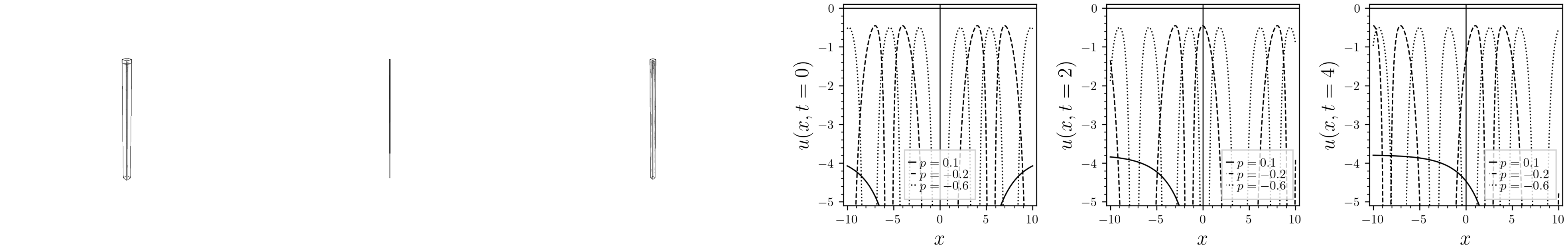
Figure 14: Spacetime evolutions of the plane periodic solution $u_{6,\text{ext gen}}$ for $c = 2$ and $p > 1$.



(a): $u_1 : |x,t| \leq 4, p = 0.1$ (b): $u_1 : |x,t| \leq 4, p = -0.2$ (c): $u_1 : |x,t| \leq 4, p = -0.6$ (d): $u_1 : |x| \leq 10, t = 0$ (e): $u_1 : |x| \leq 10, t = 2$ (f): $u_1 : |x| \leq 10, t = 4$

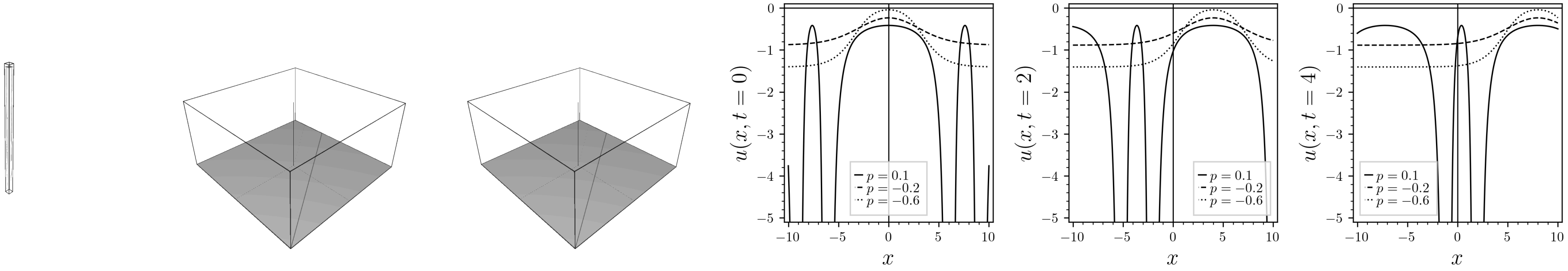
Figure 15: Spacetime evolutions of the soliton solution $u_{1,\text{ext gen}}$ for $c = 2$ and $p < 1$.

3.6. Playing with the parameter p



(a): $u_3 : |x, t| \leq 15, p = 0.1$ (b): $u_3 : |x, t| \leq 15, p = -0.2$ (c): $u_3 : |x, t| \leq 15, p = -0.6$ (d): $u_3 : |x| \leq 10, t = 0$ (e): $u_3 : |x| \leq 10, t = 2$ (f): $u_3 : |x| \leq 10, t = 4$

Figure 16: Spacetime evolutions of the non-soliton traveling wave solution $u_{3,\text{ext gen}}$ for $c = 2$ and $p < 1$.



(a): $u_6 : |x, t| \leq 1000, p = 0.1$ (b): $u_6 : |x, t| \leq 1000, p = -0.2$ (c): $u_6 : |x, t| \leq 1000, p = -0.6$ (d): $u_6 : |x| \leq 10, t = 0$ (e): $u_6 : |x| \leq 10, t = 2$ (f): $u_6 : |x| \leq 10, t = 4$

Figure 17: Spacetime evolutions of the plane periodic solution $u_{6,\text{ext gen}}$ for $c = 2$ and $p < 1$.

3.6. Playing with the parameter p

For $0 \leq p \leq 1$

- As $p \to 0$, soliton u_1 widens and amplitude decreases, indicating energy delocalization and transition towards plane-wave-like state
- Traveling wave u_3 widens with decreased depth, making localized features less pronounced and potentially tending towards constant solution
- Smaller p values suggest instability or energy dispersion, with forcing effects becoming dominant
- Solutions lose distinct characteristics and approach dissipated states

3.6. Playing with the parameter p

For $p > 1$

- Soliton u_1 becomes narrower with increased amplitude, signifying energy concentration and sharply peaked waves
- Non-soliton wave u_3 develops deeper, narrower valleys with more pronounced localized features
- Plane periodic wave u_6 shows increased oscillation amplitude and more pronounced periodic variations
- Corresponds to stronger nonlinearity or different dispersive properties controlled by p via forcing term

3.6. Playing with the parameter p

For $p < 0$

- Classical soliton loses single-hump shape, becoming oscillatory and no longer fitting classical soliton definition
- Non-soliton traveling waves transform into oscillatory patterns with complex behavior
- Plane periodic solutions remain periodic but with significantly altered waveforms, often featuring sharper characteristics and additional oscillations
- Solutions fundamentally different from $p \geq 0$ cases, potentially describing entirely different physical phenomena or mathematical structures

4. Conclusions and recommendations

4.1. Conclusions

Key contributions include

- **Methodological advancement**. Developed generalized tanh-function method with tunable parameter p and extended it with negative powers in series solution
- **New solution families**. Derived 8 unique families of exact solutions including tunable solitons, non-soliton traveling waves, and plane periodic solutions
- **Forcing function insight**. Solutions satisfy forced Boussinesq equation when $p \neq 1$, with forcing term $F(Y_p)$ dependent on parameter p

4.1. Conclusions

- **Parameter control mechanism**. Parameter p provides powerful control over solution characteristics including amplitude, width, wavelength, and fundamental form
- **Expanded solution space**. Significantly broadened known analytical solution space for Boussinesq-type equations

4.1. Conclusions

Observed effects of parameter p

- $0 \leq p \leq 1$. Decreasing p produces wider, flatter localized waves and structurally modulated periodic waves
- $p > 1$. Solutions become narrower and more sharply peaked
- $p < 0$. Fundamental transformation from hyperbolic to trigonometric character, creating diverse oscillatory patterns
- $p = 1$. Forcing term vanishes, retrieving known standard Boussinesq equation solutions

4.2. Recommendations

For immediate applications

- **Broader equation coverage**. Apply method to KdV-type equations, nonlinear Schrödinger equations, and higher-dimensional systems
- **Parameter space exploration**. Investigate complex values of p and their mathematical properties
- **Physical realizability**. Study stability and physical meaning of solutions for $p < 0$ or complex p

4.2. Recommendations

For advanced investigations

- **Analyze forcing function**. Find conditions where $F(Y_p)$ vanishes beyond $p = 1$ and explore physical interpretation of forcing terms
- **Generalize using Riccati equation**. Develop unified framework encompassing various tanh-based methods as special cases
- **Approximate solutions**. Leverage tunable parameter for constructing optimized approximations in intractable cases

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