Thesis proposal

Novel exact solutions for forced Boussinesq and Boussinesq-type equations via generalized extended tanh-function method

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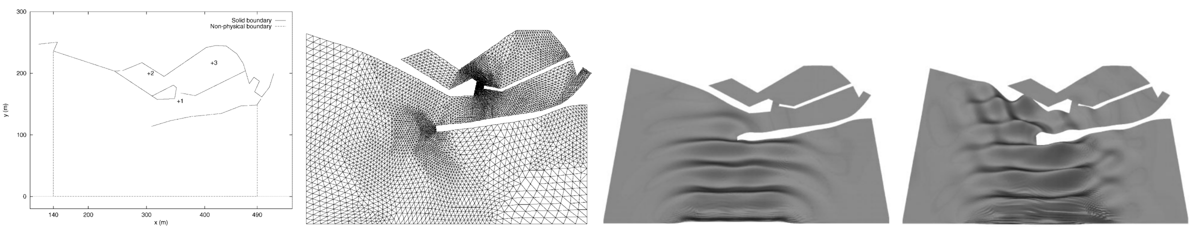
1 Introduction

Among the many nonlinear partial differential equations, one of most physically significant and mathematically interesting is the Boussinesq equation given in dimensionless form by

$$\partial_t^2 u - c^2\partial_x^2 u - \alpha\partial_x^2 u^2 - \beta\partial_x^4 u = 0, \tag{1.1}$$

which describes long waves in shallow water [1, 2] (see also [3]). While it found its primary application in hydrodynamics and coastal engineering, such as in modeling wave interactions in surf zones [4, 5], in nearshore zones [6], in swash zones [7], of irregular wave trains [8, 9], with uneven beds [10], with porous and reef-like beds [11], inside harbors [7], on bounded grids [12, 13], and on unstructured mesh [14–16] as in Fig. 1, this equation has proven surprisingly versatile. It appears in wide ranging physical systems such as nonlinear magnetosound waves in plasmas [17, 18], observed thin turbulent layers in the atmosphere [19, 20], nonlinear waves perturbations in acoustic-like regimes [21], electromagnetic waves in nonlinear dielectrics [22], elastic waves in antiferromagnets [23], and vibrations in nonlinear strings [24].



(a) Spatial domain (b) Intermediate mesh (c) Model at $t = 30$ s (d) Model at $t = 50$ s

Figure 1: Unstructured triangular meshes of the harbor geometry (1a, 1b) and simulated model of the the waves on the free surface at time t (1c, 1d) [16].

This equation derives from a family of nonlinear equations characterized by a second-order time derivative $\partial_t^2 u$ and the general form

$$\partial_t^2 u - \partial_x^2 u + P(u) = 0, \tag{1.2}$$

where $u = u(x, t)$ is a differentiable function of space x and time t, and $P(u)$ is a nonlinear term. Unlike the unidirectional Korteweg-de Vries (KdV) and KdV-type equations involving $\partial_t u$, this equation exhibits bidirectional wave propagation, traveling in both left and right directions [25]. However, despite this distinction, this equation reduces to the KdV equation by neglecting the interaction of the opposing waves and considering only one direction [26]. Moreover, the Boussinesq equation itself can be obtained from the Kadomtsev-Petviashvili (KP) equation through dimensional reduction in a moving frame [27]. Interestingly, the relationship reverses in near unidirectional propagation within the original dimensions, with the Boussinesq equation reducing to the KP equation [28]. Furthermore, for complex-valued amplitudes in the slow modulation regime, the Boussinesq equation approximates the nonlinear Schrödinger (NLS) equation. Notably, the rational solutions of the former bear resemblance to the rogue wave solutions of the latter [29]. These highly unpredictable waves, characterized by extreme localized amplitude, have garnered significant interest recently (see [29–31]). While possessing distinct physical applications, all of the aforementioned equations are interconnected within a broader family of partial differential equations describing solitary waves, as illustrated in Fig. 2. Thusly, an investigation on any of the equations could indirectly contribute to advancements in related equation families.

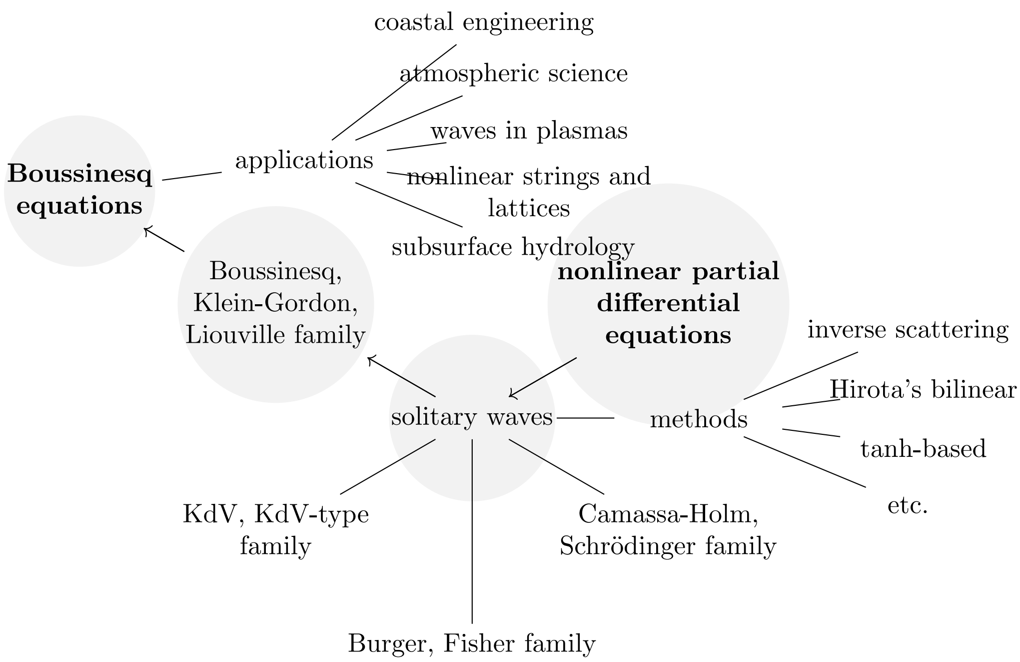


Figure 2: Conceptual map of the Boussinesq equations

Moreover, this equation incorporates two competing effects: nonlinearity represented by $P(u) = -\alpha\partial_x^2 u^2 - \beta\partial_x^4 u$ which steepens the wave, and linear dispersion embedded in dispersion relation ω which spreads the wave [2]. Unlike the classical shallow-water equations derived from the Navier-Stokes equations, Boussinesq equation accounts for frequency dispersion enabling it to model a wider range of wave phenomena, including those with smaller wavelengths [32]. The balance between nonlinearity and dispersion gives rise to soliton solutions. These are particle-like waves characterized by their single-humped profile, finite amplitude, and constant shape and speed [33]. While solitons are generally stable, certain anomalous behaviors such as the formation of singularities in a finite time and decay under perturbations have been observed in soliton solutions specific to the Boussinesq equation [27, 34, 35] (see also [36]).

Mathematically, (1.1) can admit any coefficients $\alpha, \beta, c \in \mathbb{R}$ with $\alpha > 0$, and should yield equivalent equations after rescaling and variable translation, as long as the sign of β is taken into account (otherwise, the sign does not matter if the coefficients are complex) [37]. With $\beta = 1$, the equation is *ill-posed* which means that the initial value problem cannot be solved for arbitrary data [33]. Conversely, $\beta = -1$ leads to a *well-posed* equation [38]. Despite the posedness, both *classical* Boussinesq equations admit inverse scattering formalism and are completely integrable [24, 38], making Boussinesq one of the few completely integrable equations. The former reduces the solutions of the equation to a linear integral equation and often implies the latter [37]. The latter entails an infinite number of independent conversation laws and symmetries, along with the existence of multi-solition solutions, among other notable properties [39]. Note that another family of equations known as *improved* Boussinesq equations is also being thoroughly studied, which includes a mixed fourth-order derivative $\partial_t^2\partial_x^2 u$ instead of the purely spatial ∂_x^4 term found in the classical family. This modification improves the dispersive properties [33] broadening Boussinesq equations' applicability to other wave phenomena. This family is well-posed [40, 41] albeit not completely integrable [42, 43] but will nonetheless be considered in this paper.

A variety of methods have been developed to solve the Boussinesq equation and other solitary wave-describing families of nonlinear partial differential equations. These include powerful techniques that directly deal with the partial differential equations such as inverse scattering transform [44], Bäcklund transform [45], and Hirota's bilinear method [46–48]. However, simpler methods such as direct integration

[49], homogeneous balance method [50], sine-Gordon expansion [51, 52], and tanh-function method [53, 54] have also proven effective in obtaining exact and analytic solutions. These methods capitalize on the straightforward nature of hyperbolic and exponential functions to model traveling waves, and of trigonometric functions to represent periodic waves, which solitary wave-describing equations readily accommodate. By adopting a traveling wave frame of reference, the partial differential equation is transformed into an ordinary differential equation from which closed-form solutions in terms of these transcendental functions are sought [25, 33].

Due to simplicity, the original tanh-based method has since been extended and modified in certain directions to obtain more exact traveling wave solutions. This includes the coth extension [55, 56], hyperbolic-function generalization [57, 58], trigonometric [25, 59] and exponential [60] reformulations, and generalizations to Riccati equation expansion [61, 62] and projection [63–66]. The lattermost method can obtain new families of exact solutions including non-traveling wave soliton-like ones among others.

In this paper, we formulate a generalization of the tanh-function method for solving a broad class of nonlinear partial differential equations. We then propose a further extension of this generalization. Finally, we implement both methods to obtain new tunable soliton, periodic, and other traveling wave solutions for the classical and improved Boussinesq equations, high-order Boussinesq equations, and some physically relevant variants.

This paper lays the groundwork for a robust and compact generalization of the tanh-function method. While this method may not produce multiple-soliton solutions as some existing approaches do, it has the potential to find other physically interesting and potentially unique solutions, including bright and dark solitons, complex solitons, and non-traveling wave soliton-like solutions. The focus of this paper is on developing this generalization of said method and applying it to the Boussinesq and Boussinesq-type equations. It is not intended to improve the equations themselves, although the forcing functions we derive could be considered an improvement. Furthermore, the equations reside in the $(1 + 1)$-dimension, which is both sufficient for physical purposes and consistent with literature, rather than $(2 + 1)$.

Future research could explore applying this generalization to other nonlinear systems, including those with more sophisticated properties and even approximate solutions in cases where exact solutions are elusive. One promising avenue involves either formulating a standard scheme to incorporate forcing functions directly into the nonlinear equations or devising a strategy in the derivations to eliminate the need for such functions. Additionally, formulating a more general method inspired by Riccati equation-based methods, which would encompass tanh-based methods as a special case, is an interesting direction for further investigation.

2 Methodology

In this section, we formulate our proposed generalization of the tanh-function method along with an extension of this generalization with the help of a more general and encompassing ansatz We then discuss how to obtain a new set of tunable soliton, periodic, and other traveling wave solutions for forced Boussinesq and Boussinesq-type equations.

2.1 Standard tanh method

We follow the standard tanh method for solving pdes introduced by Malfliet in [53, 54] which employs the transformation

$$u(x,t) \to U(\xi), \quad \xi = \mu(x - ct), \tag{2.1}$$

for arbitrary real constants c and μ which are usually wave speed and wave number, respectively, and the introduction of the new function

$$Y(\xi) = \tanh \xi, \tag{2.2}$$

which was specifically chosen due to a convenient property of the said function. That is, when differentiated repeatedly, tanh assumes slight variants of itself and transforms to sech quite easily as in

$$d_\xi Y = \operatorname{sech}^2 \xi = 1 - Y^2, \quad d_\xi^2 Y = -2Y + 2Y^3, \quad d_\xi^3 Y = -2 + 8Y^2 - 6Y^4, \quad \ldots . \tag{2.3}$$

Here, we find a set of algebraic functions representing various orders of derivatives. This leads to the simple yet powerful ansatz

$$U = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{2.4}$$

where coefficients a_k are real constants to be determined, and M is a positive integer extractable via balancing terms and derivatives (elaborated below).

This method includes (1) transforming the pde into a nonlinear ode, (2) balancing the highest order nonlinear term with the highest order derivative, (3) deriving and solving a nonlinear system of equations for coefficients and parameters, and (4) substituting the solutions for these coefficients and parameters back into the nonlinear ode.

But since we introduce a new independent variable inspired by a novel ansatz first introduced in [67], we expand upon the above methodology, perform a set of substitutions, and employ tricks where convenient, all within this paper's proposed modified tanh method procedure as illustrated in Fig. 3. We extend this generalization, then apply both procedures to various Boussinesq and Boussinesq-type equations.

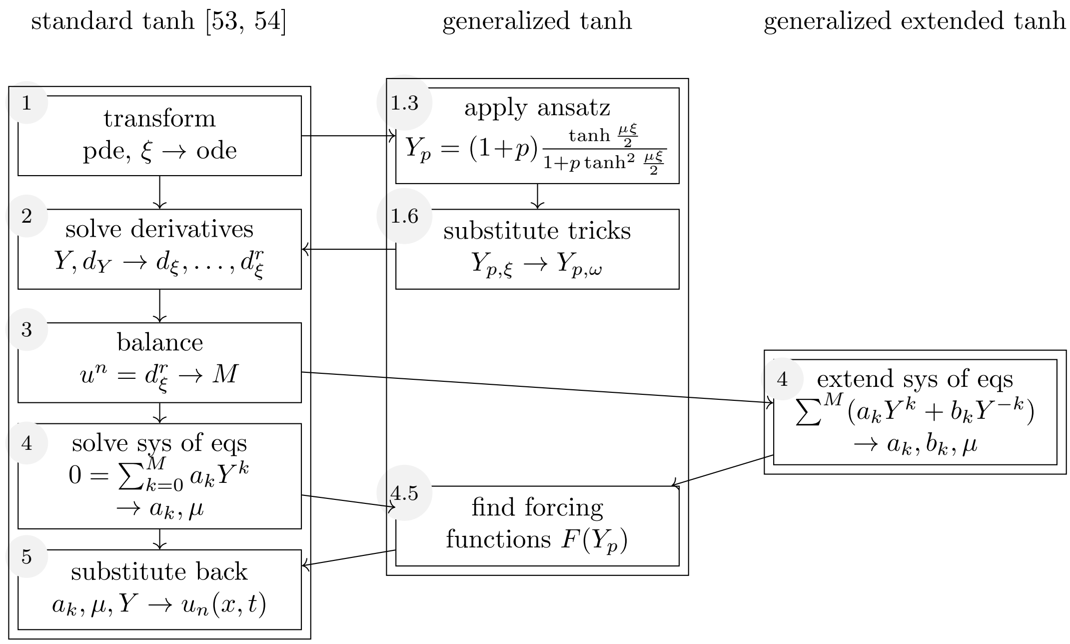


Figure 3: Procedural diagrams of the standard and proposed methods

2.2 Generalized tanh method

We formulate a modification of the tanh method based on the new introductory function as ansatz

$$Y_{p,\xi} = Y_p(\mu\xi) = (1+p)\frac{\tanh\dfrac{\mu\xi}{2}}{1+p\tanh^2\dfrac{\mu\xi}{2}}, \quad 0 \le p \le 1, p \in \mathbb{R}. \tag{2.5}$$

The key feature of this ansatz is the tunable parameter p. It could allow for solutions to be either adaptively tailored to the specific problem at hand or precisely fine-tuned to meet specific conditions.

Using the variable $\xi = x - ct$, we transform any pde into an ode as in

$$\begin{aligned} &P(u, \partial_t u, \partial_x u, \partial_t^2 u, \partial_x^2 u, \partial_t^3 u, \ldots) = 0 \\ \implies &Q(U, d_\xi U, d_\xi^2 U, d_\xi^3 U, \ldots) = 0 \end{aligned} \tag{2.6}$$

together with their respective solutions $u(x,t)$ and $U(\xi)$. Assuming the integration constants vanish, we iteratively integrate this ode until desired, say until

$$\tilde{Q}(U, d_\xi U, d_\xi^2 U, d_\xi^3 U, \ldots) = 0, \tag{2.7}$$

as long as all terms retain derivatives. We then compute for the higher-order derivatives

$$d_\xi, d_\xi^2, d_\xi^3, \dots, d_\xi^n, \tag{2.8}$$

where n represents the highest order of differentiation present in the integrated ode (2.7). Note that this computation is particularly cumbersome.

Next, we still assume that the use of the finite series expansion in (2.4) as a solution to the ode such that $u(x,t) = U(\xi) = S(Y)$ is admissible under the tanh method. Substituting this series into the ode yields a semblance of algebraic equation in powers of Y which follow the mapping

$$u \to M, \quad u^2 \to 2M, \quad \dots, \quad u^n \to nM, \tag{2.9a}$$

$$\partial u \to M+1, \quad \partial^2 u \to M+2, \quad \dots, \quad \partial^r u \to M+r. \tag{2.9b}$$

We employ this to balance the highest order nonlinear term with the highest order derivative in the integrated ode (2.7) and determine the balance constant M to use in (2.4). We reject any non-positive integer M and adjust in the integration step accordingly. If inconvenient, we apply the transformation

$$U(\xi) = \phi^M \xi, \tag{2.10}$$

then substitute it back and attempt to determine M again as long as M is a fraction or a negative integer as suggested in [65].

Then, we substitute the necessary derivatives (2.8) and the series (2.4) with determined M into the integrated ode (2.7), and group the terms according to their powers in Y. For terms with non-integral powers of Y, we introduce different forcing functions $F(Y)$ that make them vanish such that we have

$$\tilde{Q}(U, d_\xi U, d_\xi^2 U, d_\xi^3 U, \dots; Y) + F(Y) = 0. \tag{2.11}$$

To be consistent for all values of Y, the coefficient expressions must each equate to zero. This results in a nonlinear system of algebraic equations for the mathematical coefficients a_n for $n \geq 0$, $n \in \mathbb{Z}$ and physical coefficients such as the wave number μ. We then solve this system by hand, and utilize a computer algebra system such as the free and open-source SageMath for tedious calculations when needed.

Finally, we substitute the determined solutions for the coefficients and parameters back into the integrated ode, apply restricting conditions where necessary, and obtain a set of tunable soliton and plane periodic solutions.

2.3 Generalized extended tanh method

In the previous method, we only have algebraic terms in positive powers of Y in the finite series (2.4) to restrict the solution space to tanh and sech-employing solutions. We remove this limitation to explore additional set of solutions, particularly coth and csch-based solutions, by extending the series into

$$S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}. \tag{2.12}$$

as inspired by an extension of the tanh method in [55, 56].

2.4 Boussinesq equations

After we formulate our proposed generalization of the tanh-function method along with an extension of this generalization, we implement both methods to obtain new tunable solutions to the following forms of the Boussinesq and Boussinesq-type equations: classical and improved Boussinesq equations with $\beta = \pm 1$ [1, 2, 33]

$$\partial_t^2 u - c^2 \partial_x^2 u - \alpha \partial_x^2 u^2 - \beta \partial_x^4 u = 0, \tag{2.13a}$$

$$\partial_t^2 u - c^2 \partial_x^2 u - \alpha \partial_x^2 u^2 - \beta \partial_t^2 \partial_x^2 u = 0, \tag{2.13b}$$

generalized Boussinesq systems [42]

$$\partial_t^2 - c_0^2 \partial_x^2 u - \partial_x^2 \left(F(u) - \beta \partial_x^2 u + \partial_x^4 u \right) = 0, \tag{2.14a}$$

$$\partial_t^2 - \partial_x^2 \left(u + F(u) - \beta \partial_t^2 u \right) + \beta \partial_t^4 u = 0, \tag{2.14b}$$

high-order modified Boussinesq [65, 68]

$$\partial_t^2 + \alpha\partial_t\partial_x^2 u + \beta\partial_x^4 u + r\partial_x^2 u^n = 0 \tag{2.15}$$

variant Boussinesq equations [61, 69]

$$\partial_t u + \partial_x v + u\partial_x u + p\partial_t\partial_x^2 u = 0, \tag{2.16}$$

$$\partial_t v + \partial_x uv + q\partial_x^3 u = 0, \tag{2.17}$$

dissipative Boussinesq equation [70]

$$\partial_t \eta + h\partial_x u = \frac{h^3}{6}\partial_x^3 u, \tag{2.18a}$$

$$\partial_t u + g\partial_x \eta + \delta_1 u = \frac{1}{2}\delta_1 h^2 \partial_x^2 u + \frac{1}{2}h^2\partial_t\partial_x^2 u, \tag{2.18b}$$

$$\partial_t u + g\partial_x \eta = \delta_2 \partial_x^2 u + \frac{1}{2}h^2\partial_t\partial_x^2 u. \tag{2.18c}$$

the Boussinesq-type equation that describes vibration in magneto-electro-elastic circular rod [71]

$$\partial_t^2 u - v_0^2\partial_x^2 u - \partial_x^2\left(\frac{v_0^2}{2}u^2 + m\partial_t^2 u\right) = 0, \tag{2.19}$$

and possibly others that are physically relevant to plasma physics and atmospheric science.

3 Expected results

We note that for the classical Boussinesq equation (1.1) with $c = 3$, $\beta = -1$, there are two soliton and two traveling wave solutions

$$u_1(x,t) = \frac{c^2-1}{2}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c^2-1}(x-ct)\right), \qquad c^2 > 1, \tag{3.1a}$$

$$u_2(x,t) = -\frac{c^2-1}{6}\left[1 - 3\tanh^2\left(\frac{1}{2}\sqrt{1-c^2}(x-ct)\right)\right], \qquad c^2 < 1, \tag{3.1b}$$

$$u_3(x,t) = -\frac{c^2-1}{2}\,\mathrm{csch}^2\left(\frac{1}{2}\sqrt{c^2-1}(x-ct)\right), \qquad c^2 > 1, \tag{3.1c}$$

$$u_4(x,t) = -\frac{c^2-1}{6}\left[1 - 3\coth^2\left(\frac{1}{2}\sqrt{1-c^2}(x-ct)\right)\right], \qquad c^2 < 1, \tag{3.1d}$$

with corresponding plane periodic solutions

$$u_5(x,t) = \frac{c^2-1}{2}\sec^2\left(\frac{1}{2}\sqrt{1-c^2}(x-ct)\right), \qquad c^2 < 1, \tag{3.2a}$$

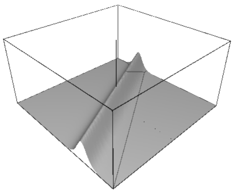
$$u_6(x,t) = -\frac{c^2-1}{6}\left[1 - 3\tan^2\left(\frac{1}{2}\sqrt{c^2-1}(x-ct)\right)\right], \qquad c^2 > 1, \tag{3.2b}$$

$$u_7(x,t) = \frac{c^2-1}{2}\csc^2\left(\frac{1}{2}\sqrt{1-c^2}(x-ct)\right), \qquad c^2 < 1, \tag{3.2c}$$

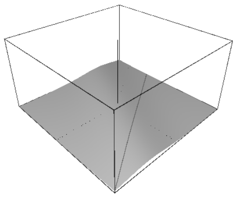
$$u_8(x,t) = -\frac{c^2-1}{6}\left[1 - 3\cot^2\left(\frac{1}{2}\sqrt{c^2-1}(x-ct)\right)\right], \qquad c^2 > 1, \tag{3.2d}$$

all of which are obtainable via an extension of the tanh method in [25]. Now, say, the other Boussinesq and Boussinesq-type equations also have a certain set of solutions depending on the methods used.

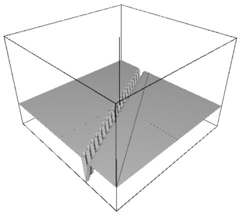
In this paper, we shall obtain a new set of soliton, periodic, and other traveling wave solutions for the forced variants of these Boussinesq and Boussinesq-type equations. Given the existence of these known solutions obtained through other tanh and non-tanh-based methods, we expect our proposed generalized extended tanh method will produce at least as many, if not more, solution families. Ideally, the new solutions would exhibit similar or somewhat similar behavior to the ones presented above. That is, their analytical behavior resembles the properties seen in Fig. 4. In other words, the new solutions should at least be analytic if not traveling wave solutions, but ideally both. Furthermore, we anticipate that for specific parameter values, the solutions obtained by our method reduce back to the standard tanh method solutions.



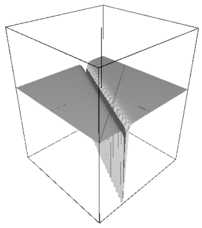
(a) $u_1 : c = 2$,
$|x, t| \leq 5$



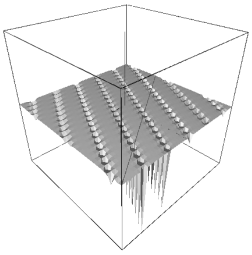
(b) $u_2 : c = \frac{1}{2}$,
$|x, t| \leq 5$



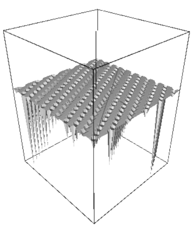
(c) $u_3 : c = 2$,
$|x, t| \leq 20$



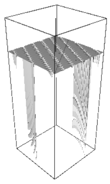
(d) $u_4 : c = \frac{1}{2}$,
$|x, t| \leq 20$



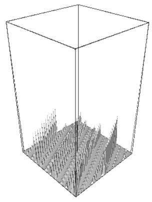
(e) $u_5 : c = \frac{1}{2}$,
$|x, t| \leq 1000$



(f) $u_6 : c = 2$,
$|x, t| \leq 1000$



(g) $u_7 : c = \frac{1}{2}$,
$|x, t| \leq 1000$



(h) $u_8 : c = 2$,
$|x, t| \leq 1000$

Figure 4: Graphs of the solutions to the classical Boussinesq equation via tanh method

4 Timeline

In this section, we outline the research project's structure across a two-month semester and two five-month semesters as detailed in Figs. 5, 6 and 7. The project involves two core tasks (solving problems and writing weekly reports) and several tasks distributed across six milestones. Three milestones serve as semestral checkpoints and require progress report presentations to the panel. The remaining milestones encompass this proposal and two written deliverables: a manuscript for the home department and a publishable article for the broader scientific audience.

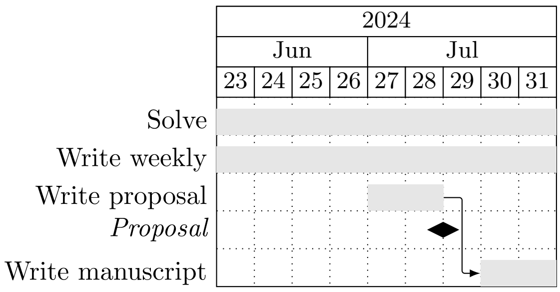


Figure 5: Gantt chart for Year 2024 Semester 0

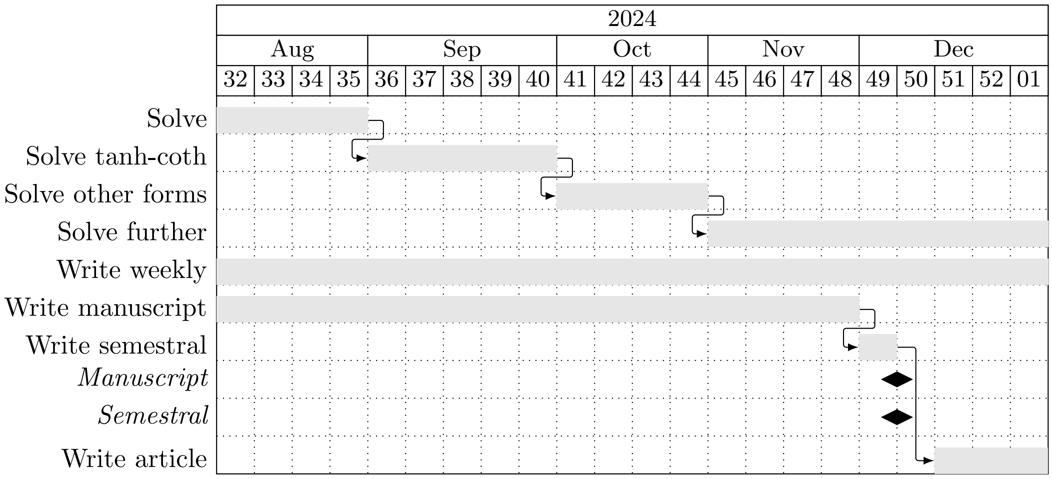


Figure 6: Gantt chart for Year 2024 Semester 1

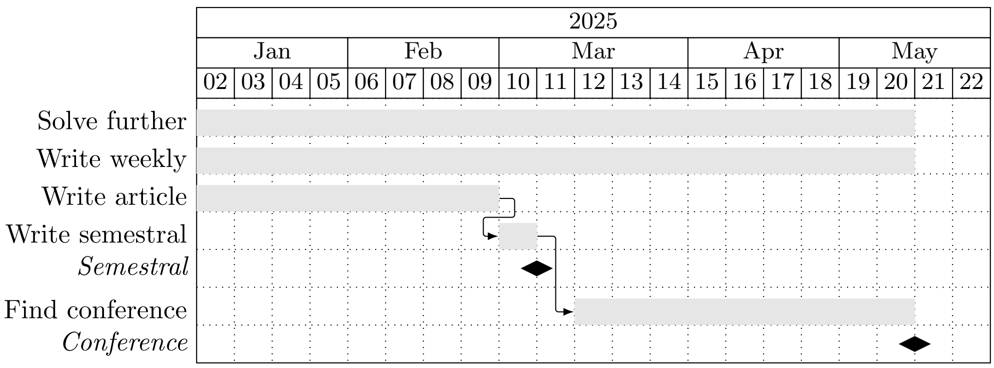


Figure 7: Gantt chart for Year 2024 Semester 2

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